Blind Receivers based on Tensor Decompositions. Application in DS-CDMA and over-sampled systems.

Dimitri Nion and Lieven De Lathauwer

***Abstract*— In this paper, we propose a survey on Blind Receivers based on Tensor Decompositions in Block Components. In the uplink, if the receiver is equipped with an antenna array, the spatial, temporal and CDMA code or over-sampling diversities allow to model the chip-rate sampled received signal as a third-order tensor. Each user's contribution is then blindly estimated by decomposition of this tensor of observations. As we will show, different propagation scenarios require different tensor decompositions. We will then briefly address the algorithmic aspect to compute these tensor decompositions.**

I. Introduction

Let us consider R users transmitting with a single antenna, at the same time within the same bandwidth, frames of J symbols towards an array of K antennas with unknown geometry. The channel is supposed to be stationary over the interval of duration $J.T_s$, where T_s is the symbol-period. We denote by $\mathbf{s}_r = [s_{1r} s_{2r} \ldots s_{Jr}]$, the symbol sequence of user r. From the observations given by the antenna array, we wish to estimate each user's symbol sequence in a blind way, i.e., we do not use training sequences.

If we over-sample the signal received by each antenna by a factor I, i.e., we collect I samples within each symbol period, we finally get a set of IJK samples that can be arranged in a third-order tensor $\mathcal{Y} \in \mathbb{C}^{I\times J\times K}$. Each dimension of this observation tensor corresponds to an available diversity. The blind problem is then solved by the decomposition of $\mathcal{Y}$ as

$$\mathcal{Y} = \sum_{r=1}^{R} \mathcal{Y}_r, \tag{1}$$

where $\mathcal{Y}_r \in \mathbb{C}^{I\times J\times K}$ fully characterizes the global contribution of user r.

The models of this paper work both for DS-CDMA and over-sampled systems. We do not assume knowledge of the spreading codes or pulse shape filters, not even for a user of interest. The deterministic blind receivers we propose rather exploit the algebraic structure of $\mathcal{Y}$. However, this structure is not the same according to the propagation scenario. For instance, the PARAFAC decomposition of $\mathcal{Y}$ is the solution in a single-path scenario, while Block-Component-Decompositions (BCD) are needed for more complex channels.

In Section 2, we derive the analytic expression for the transmitted signal in CDMA and over-sampled systems. In Section 3, we associate a specific tensor decomposition to the received signal for each of the three propagation scenarios under consideration. In Section 4, we give references to several algorithms that can be used to compute the decompositions.

II. Transmitted signals

A. DS-CDMA system

We denote by I the spreading factor, i.e., $T_s = I.T_c$, where T_c is the chip period. The spreading waveform $e_r(t)$ of user r is built by modulation of his spreading sequence $\mathbf{c}_r = [c_{1r} \ldots c_{Ir}]$ by a pulse-shape (raised-cosine) filter $g_{r,T_c}(t)$:

$$e_r(t) = \sum_{i=1}^{I} g_{r,T_c}(t - iT_c) c_{ir}.$$

Note that $g_{r,T_c}(t)$ is indexed by r since the technique we propose does not require the same pulse-shape filter for each user. The index T_c means that the width of the main lobe of this filter is $2T_c$. The baseband signal $x_r(t)$ transmitted by user r is:

$$\begin{aligned} x_r(t) &= \sum_{j=1}^{J} s_{jr} e_r(t - jT_s) \\ &= \sum_{j=1}^{J} s_{jr} \sum_{i=1}^{I} g_{r,T_c}(t - iT_c - jT_s) c_{ir}. \end{aligned} \tag{2}$$

Part of this research was carried out when the authors were with Lab. ETIS, UMR 8051, 6, avenue du Ponceau, 95014 Cergy-Pontoise Cedex, France. D. Nion now holds a post-doc position in department ECE, Technical University of Crete, Greece (e-mail: nion@telecom.tuc.gr, tel: +30-28210-37248). L. De Lathauwer is now with the Research Group ESAT-SCD, K.U.Leuven, Kasteelpark Arenberg 10, B-3001 Leuven-Heverlee, Belgium (delathau@esat.kuleuven.be) and with the K.U.Leuven Campus Kortrijk, Subfaculty Sciences, E. Sabbelaan 53, 8500 Kortrijk, Belgium (lieven.delathauwer@kuleuven-kortrijk.be).

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B. Over-sampled system

In this system, $\mathbf{s}_r$ is directly modulated (without code spreading) by a pulse-shape filter $g_{r,T_s}(t)$ defined at the symbol rate. The baseband signal $x_r(t)$ transmitted by user r is

$$x_r(t) \quad = \quad \sum_{j=1}^{J} s_{jr} g_{r,T_s}(t - jT_s). \tag{3}$$

In this system, an extra diversity will be created by temporally over-sampling the received signals. In this case, I does not stand for the spreading factor as in CDMA, but is called the over-sampling factor. We will however keep the same notation: $T_s = I.T_c$.

III. Analytic and algebraic models for received signals

In this section, we consider three propagation scenarios and for each scenario, we associate the analytic expression for the received signal to its algebraic equivalent.

A. Memoryless Channel

1) Analytic Model: We suppose that each of the signals $x_r(t)$, $r = 1 \ldots R$ are received via a single path characterized by a fading-factor β_r, an angle of arrival θ_r and a delay τ_r that holds propagation delay and asynchronism. The baseband signal $y_k(t)$ received by antenna k is:

$$y_k(t) = \sum_{r=1}^{R} \beta_r a_k(\theta_r) x_r(t - \tau_r), \tag{4}$$

where $a_k(\theta_r)$ is the response of antenna k to the angle θ_r.

For both CDMA and over-sampled systems, the sample y_{ijk} of $y_k(t)$ at sampling instant $t = (jI + i)T_c$, can be written as:

$$y_{ijk} \quad = \quad \sum_{r=1}^{R} a_{kr} s_{jr} h_{ir}, \tag{5}$$

where $a_{kr} = a_k(\theta_r)$. For a DS-CDMA system, $h_{ir} = \beta_r c_{ir} g_{r,T_c}(t - iT_c - jT_s - \tau_r) \mid_{t=iT_c+jT_s}$ is the sample of the global channel at instant $t = (jI + i)T_c$. Note that if $\tau_r = 0$, then $h_{ir} = \beta_r c_{ir}$. For an over-sampled system, $h_{ir} = \beta_r g_{r,T_s}(t - jT_s - \tau_r) \mid_{t=iT_c+jT_s}$.

2) Algebraic Model: PARAFAC: Sidiropoulos, Giannakis and Bro were the first to use a multilinear algebra point of vue in wireless communications in 2000. In fact, they have shown that the analytic model of Eq. (5) is a PARAFAC decomposition of the tensor of observations $\mathcal{Y} \in \mathbb{C}^{I \times J \times K}$ holding the entries y_{ijk} [1].

Parallel Factor Analysis (PARAFAC) was introduced in [2], [3] and reintroduced in [4], [5]. It is a powerful technique to decompose a rank-R tensor in a linear combination of R rank-1 tensors. Algebraically, the PARAFAC decomposition of $\mathcal{Y}$ is written as

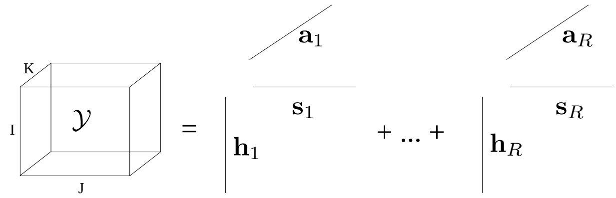

Fig. 1. Schematic representation of the PARAFAC decomposition

$$\mathcal{Y} = \sum_{r=1}^{R} \mathbf{h}_r \circ \mathbf{s}_r \circ \mathbf{a}_r, \tag{6}$$

where $\mathbf{h}_r \in \mathbb{C}^I$, $\mathbf{s}_r \in \mathbb{C}^J$ and $\mathbf{a}_r \in \mathbb{C}^K$ hold the samples h_{ir}, s_{ir} and a_{ir}, respectively, and $\circ$ is the outer product [6]. This trilinear decomposition is visualized in Fig. 1.

B. Far-Field reflections

We now consider a multipath propagation scenario where the reflectors are only located in the far-field, from the receiver point of vue. This assumption means that the angular spread between all paths incoming from the same user is negligible. However, the delay spread is such that Inter-Symbol-Interference (ISI) might occur.

1) Analytic Model: For user r, we denote by $h_r(t)$ the global Channel Impulse Response.

- For CDMA systems, $h_r(t)$ results from convolution between the (finite) impulse response of the effective propagation channel and the spreading waveform $e_r(t)$.

- For over-sampled systems, $h_r(t)$ results from convolution of the same impulse response by the pulse-shape filter $g_{r,T_s}(t)$.

Let $L_r T_s$ be the duration of $h_r(t)$, meaning that ISI occurs on L_r consecutive symbols. The sample y_{ijk} of the signal received by antenna k at chip instant $(jI + i)T_c$ can be written as:

$$y_{ijk} = \sum_{r=1}^{R} a_k(\theta_r) \sum_{l=1}^{L_r} h_r(i + (l-1)I) s_{j-l+1,r}, \tag{7}$$

where $a_k(\theta_r)$ is the response of antenna k to the (mean) angle of arrival θ_r, and where $h_r(i + (l-1)I)$ is the sample of $h_r(t)$ at instant $(i + (l-1)I)T_c$.

2) Algebraic Model: BCD-(L_r,L_r,1): The analytic model of Eq. (7) can equivalently be written as the Block Component Decomposition of $\mathcal{Y}$ in rank-(L_r,L_r,1) terms [7]–[11]. This decomposition, referred to as BCD-(L_r,L_r,1), is a generalization of PARAFAC in the sense that each contribution $\mathcal{Y}_r$ now results from two rank-L_r matrices $\mathbf{H}_r \in \mathbb{C}^{I \times L_r}$ and $\mathbf{S}_r \in \mathbb{C}^{J \times L_r}$, and from one vector $\mathbf{a}_r \in \mathbb{C}^{K \times 1}$, such that

$$\mathcal{Y} = \sum_{r=1}^{R} (\mathbf{H}_r \cdot \mathbf{S}_r^T) \circ \mathbf{a}_r. \tag{8}$$

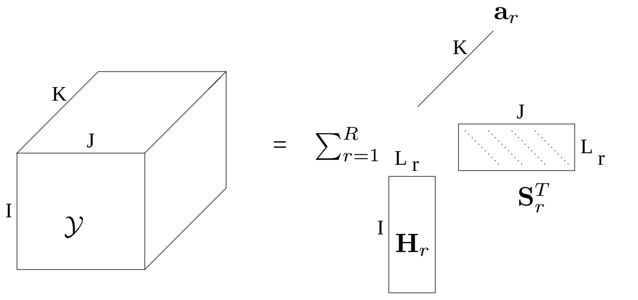

Fig. 2. Representation of the BCD-(L_r,L_r,1) with Toeplitz structure on $\mathbf{S}_r$

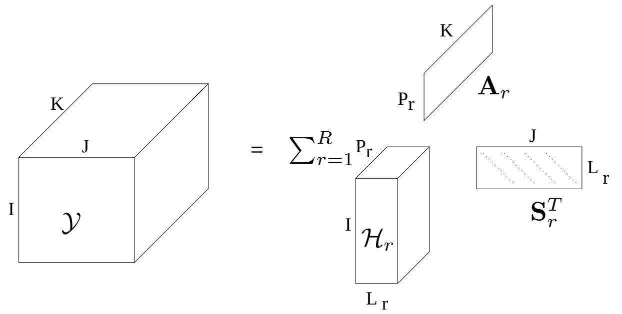

Fig. 3. Representation of the BCD-(L_r,P_r,.) with Toeplitz structure on $\mathbf{S}_r$

$\mathbf{H}_r$ holds samples of the global channel, i.e., $[\mathbf{H}_r]_{i,l} = h_r(i + (l-1))$. $\mathbf{S}_r$ has a Toeplitz structure and holds the symbols transmitted with ISI, $[\mathbf{S}_r]_{j,l} = s_{j-l+1,r}$. As for PARAFAC, $\mathbf{a}_r$ holds the coefficients $a_k(\theta_r)$. Fig. 2 is a schematic representation of the BCD-(L_r,L_r,1) terms.

C. Specular Multipath Channel

We now consider a specular multipath channel, where we associate P_r different paths to user r. The p^{th} path of user r is characterized by the triplet ($\beta_{p,r}$, $\theta_{p,r}$, $\tau_{p,r}$), where $\beta_{p,r}$ is the fading factor, $\theta_{p,r}$ is the angle of arrival and $\tau_{p,r}$ is the delay.

1) Analytic Model: For both DS-CDMA and over-sampled system, the response of the global channel between user r and antenna k can now be written as:

$$h_{k,r}(t) = \sum_{p=1}^{P_r} \beta_{p,r} a_k(\theta_{p,r}) w_r(t - \tau_{p,r}),$$

where $w_r(t-\tau_{p,r}) = e_r(t-\tau_{p,r})$ for CDMA and $w_r(t-\tau_{p,r}) = g_{r,T_s}(t-\tau_{p,r})$ for an over-sampled system. Let L_r be the length of this global channel impulse response. For both DS-CDMA and over-sampled system, y_{ijk} can then be written as

$$y_{ijk} = \sum_{r=1}^{R}\sum_{p=1}^{P_r} \beta_{p,r} a_k(\theta_{p,r}) \sum_{l=1}^{L_r} w_{p,r}(i+(l-1)I) s_{j-l+1,r}, \quad (9)$$

where $w_{p,r}(i+(l-1)I)$ is the sample of $w_r(t-\tau_{p,r})$ at instant $t = (i + (l-1)I)T_c$.

2) Algebraic Model: BCD-(L_r,P_r,.): The analytic model of Eq. (9) can equivalently be written as the Block Component Decomposition of $\mathcal{Y}$ in rank-(L_r,P_r,.) terms [7]–[9], [12]. This decomposition, referred to as BCD-(L_r,P_r,.), generalizes both PARAFAC and BCD-(L_r,L_r,.). Each contribution $\mathcal{Y}_r$ now results from a rank-L_r Toeplitz matrix $\mathbf{S}_r \in \mathbb{C}^{J\times L_r}$, with $[\mathbf{S}_r]_{j,l} = s_{j-l+1,r}$, that holds the symbols, a rank-P_r matrix $\mathbf{A}_r \in \mathbb{C}^{K\times P_r}$, with $[\mathbf{A}_r]_{k,p} = a_k(\theta_{p,r})$, that holds the response of the K antennas to the P_r paths, and from a tensor $\mathcal{H}_r \in \mathbb{C}^{I\times L_r\times P_r}$, with $[\mathcal{H}_r]_{i,l,p} = w_{p,r}(i + (l-1)I)$, that holds the coefficients of the global channel. The BCD-(L_r,P_r,.) is defined by

$$\mathcal{Y} = \sum_{r=1}^{R} \mathcal{H}_r \bullet_2 \mathbf{S}_r \bullet_3 \mathbf{A}_r, \quad (10)$$

where $\bullet_n$ is the mode-n product [6], [8]. Fig. 3 is a schematic representation of the BCD-(L_r,P_r,.).

A similar but formally different tensor-based formulation for this problem is presented in [13], [14].

IV. Algorithms

Computation of PARAFAC, BCD-(L_r,L_r,1) and BCD-(L_r,P_r,.) relies on the estimation of three unknown matrices $\mathbf{A}$, $\mathbf{S}$ and $\mathbf{H}$ of which dimensions depend on the decomposition under consideration.

For PARAFAC, $\mathbf{H} = [\mathbf{h}_1 \ldots \mathbf{h}_R]$, $\mathbf{S} = [\mathbf{s}_1 \ldots \mathbf{s}_R]$ and $\mathbf{A} = [\mathbf{a}_1 \ldots \mathbf{a}_R]$ have dimensions $(I \times R)$, $(J \times R)$ and $(K \times R)$, respectively. Let us denote $\bar{L} = \sum_{r=1}^{R} L_r$, $\bar{P} = \sum_{r=1}^{R} P_r$ and $\bar{M} = \sum_{r=1}^{R} L_r P_r$.

For BCD-(L_r,L_r,1), $\mathbf{H} = [\mathbf{H}_1 \ldots \mathbf{H}_R]$, $\mathbf{S} = [\mathbf{S}_1 \ldots \mathbf{S}_R]$ and $\mathbf{A} = [\mathbf{a}_1 \ldots \mathbf{a}_R]$ have dimensions $(I\times\bar{L})$, $(J\times\bar{L})$ and $(K\times R)$, respectively.

For BCD-(L_r,P_r,.), $\mathbf{H} = mat([\mathcal{H}_1 \ldots \mathcal{H}_R])$, $\mathbf{S} = [\mathbf{S}_1 \ldots \mathbf{S}_R]$ and $\mathbf{A} = [\mathbf{A}_1 \ldots \mathbf{A}_R]$ have dimensions $(I \times \bar{M})$, $(J \times \bar{L})$ and $(K \times \bar{P})$, respectively, where mat is an operator that stacks all entries of a tensor in a matrix.

In the application of this paper, $\mathbf{S}$ has a block-Toeplitz structure in the two block-terms decompositions and one way to achieve blind equalization within each user's contribution is to preserve this structure in all steps of the algorithms.

Several algorithms have been proposed in the literature to compute tensor decompositions. The presentation in detail of these algorithms is beyond the scope of this paper. However, we shortly adress the principle of some of these algorithms and give references where further details can be found.

A. Alternating Least Squares

The "Alternating Least Squares" (ALS) algorithm is a well-known technique to compute the PARAFAC decomposition [15], [16] and it has been extended to the decomposition of a tensor in Block Terms in [9]. This algorithm exploits the multilinearity of the algebraic model to alternate between conditional least-squares updates of the three unknown matrices in each iteration. The way ALS can be adapted to preserve the block-Toeplitz structure of $\mathbf{S}$ within each iteration is described in [10], [12]. The main drawback of ALS is its sensitivity to ill-conditioned data and near-far effect, which are known to produce swamps, i.e., many iterations with convergence speed almost null, after which convergence resumes [17], [18]. One way to reduce the length of swamps is to introduce a Line Search step before each ALS iteration.

B. Line Search

In [4], [19], Line Search was proposed to speed up convergence of ALS for PARAFAC. For a given iteration, this technique consists of the linear interpolation of the three unknown matrices from their previous estimates, after which the interpolated matrices are used as inputs of the ALS update. The challenge of Line Search is to find a “good” step size in the search directions to speed up convergence. For real-valued tensors, an “Enhanced Line Search” technique that calculates the *optimal* step size has been proposed in [20], [21]. This method has been extended to complex-valued tensors that follow PARAFAC or BCD in [22]. As a result, the length of swamps is drastically reduced at a negligible computational cost.

C. Levenberg-Marquardt

Another approach is the reformulation of the estimation problem as a classical optimization problem. In [23], a Levenberg-Marquardt (LM) algorithm is proposed for PARAFAC and it has been adapted to BCD in [24]. This algorithm is based on the well-known Gauss-Newton curve fitting technique. In contrast to ALS, the factors in the three modes are updated at the same time. As a result, this algorithm provides quadratic convergence in the final iterations and thus converges (much) faster than ALS and ALS with Line Search. Moreover, it is well adapted to separation of ill-conditioned data and small-power contributions. However, the main drawback of this algorithm is its computational cost that becomes prohibitive when the data size increases. One way to overcome this drawback is to perform a dimensionality reduction [25]–[27] of $\mathcal{Y}$ and then calculate its decomposition in the compressed space.

D. Simultaneous Diagonalization

Under some conditions on the dimensions, PARAFAC can be formulated as a problem of simultaneous diagonalization of a set of matrices [28], [29]. This results in a fast and reliable way to compute this decomposition. Moreover, this reformulation of PARAFAC involves a new bound for its uniqueness, which is much more relaxed than the Kruskal bound [30]. If the value of L_r is the same for each component, then the resulting BCD-(L,L,1) can also be expressed in terms of simultaneous diagonalization [11]. This approach also involves a new bound, much more relaxed than the one derived in [8]. The analytic expression for this new bound is being developed.

V. Conclusion

In this paper, we have shown how the blind multi-user separation-equalization problem can be solved by the decomposition of a third-order tensor, provided that the signals are received by an antenna array. This approach works both for CDMA and over-sampled systems. It does not require knowledge of antenna array geometry, neither of CDMA codes or pulse shape filters. We have shown how different propagation scenarios lead to different tensor decompositions. The latter can be calculated by several specific algorithms that have been proposed in the literature. Another important issue which is still under intensive development concerns the uniqueness of Block Component Decompositions.

References

[1] N. D. Sidiropoulos, G. B. Giannakis, and R. Bro, “Blind PARAFAC Receivers for DS-CDMA Systems,” *IEEE Trans. Signal Proc.*, vol. 48, pp. 810–823, 2000.

[2] F. L. Hitchcock, “The Expression of a Tensor or a Polyadic as a Sum of Products,” *J. Math. Phys.*, vol. 6, no. 1, pp. 164–189, 1927.

[3] ——, “Multiple Invariants and Generalized Rank of a p-way Matrix or Tensor,” *J. Math. Phys.*, vol. 7, no. 1, pp. 39–79, 1927.

[4] R. A. Harshman, “Foundations of the PARAFAC procedure: Model and Conditions for an ‘explanatory’ Multi-mode Factor Analysis,” *UCLA Working Papers in Phonetics*, vol. 16, pp. 1–84, 1970.

[5] J. D. Carroll and J. Chang, “Analysis of Individual Differences in Multidimensional Scaling via an N-way Generalization of “Eckart-Young” Decomposition,” *Psychometrika*, vol. 35, no. 3, pp. 283–319, 1970.

[6] L. De Lathauwer, “Signal Processing based on Multilinear Algebra,” Ph.D. dissertation, Faculty of Engineering, K.U. Leuven, Belgium, 1997.

[7] ——, “Decompositions of a Higher-Order Tensor in Block Terms – part I: Lemmas for Partitioned Matrices,” *SIAM J. Matrix Anal. Appl.*, 2006, accepted.

[8] ——, “Decompositions of a Higher-Order Tensor in Block Terms – part II: Definitions and Uniqueness,” *SIAM J. Matrix Anal. Appl.*, 2006, accepted.

[9] L. De Lathauwer and D. Nion, “Decompositions of a Higher-Order Tensor in Block Terms – part III: Alternating Least Squares Algorithms,” *SIAM J. Matrix Anal. Appl.*, 2006, accepted.

[10] L. De Lathauwer and A. de Baynast, “Blind Deconvolution of DS-CDMA Signals by Means of Decomposition in Rank-(1,L,L) Terms,” *IEEE Trans. Signal Proc.*, 2007, accepted.

[11] D. Nion and L. De Lathauwer, “A Tensor-Based Blind DS-CDMA Receiver Using Simultaneous Matrix Diagonalization,” in *Proc. IEEE Workshop on Signal Processing Advances in Wireless Communications (SPAWC)*, Helsinki, FIN., 2007.

[12] ——, “A Block Factor Analysis Based Receiver for Blind Multi-User Access in Wireless Communications,” in *Proc. IEEE Int. Conf. on Acoustics, Speech and Signal Processing (ICASSP)*, Toulouse, FR., 2006, pp. 825–828.

[13] A. de Almeida, G. Favier, and J. Mota, “PARAFAC-based unified tensor modeling for wireless communication systems with application to blind multiuser equalization,” *Signal Proc.*, vol. 87, pp. 337–351, 2007.

[14] ——, "Generalized parafac model for multidimensional wireless communications with application to blind multiuser equalization," in *Proc. ASILOMAR*, Oct 30 – Nov 2 2005.

[15] R. Bro, "PARAFAC: Tutorial and Applications," *Chemom. Intell. Lab. Syst.*, vol. 38, pp. 149–171, 1997.

[16] A. Smilde, R. Bro, and P. Geladi, *Multi-way Analysis. Applications in the Chemical Sciences*. Chichester, U.K.: John Wiley and Sons, 2004.

[17] P. K. Hopke, P. Paatero, H. Jia, R. T. Ross, and R. A. Harshman, "Three-way (PARAFAC) Factor Analysis: Examination and Comparison of Alternative Computational Methods as applied to ill-conditioned data," *Chemom. Intell. Lab. Syst.*, vol. 43, pp. 25–42, 1998.

[18] W. Rayens and W. Mitchell, "Two-Factor Degeneracies and a Stabilization of PARAFAC," *Chemom. Intell. Lab. Syst.*, vol. 38, pp. 173–181, 1997.

[19] R. Bro, "Multi-way Analysis in the Food Industry: Models, Algorithms, and Applications," Ph.D. dissertation, University of Amsterdam, Amsterdam, 1998.

[20] M. Rajih and P. Comon, "Enhanced Line Search: A Novel Method to Accelerate PARAFAC," in *Proc. of Eusipco'05*, 2005.

[21] M. Rajih, P. Comon, and R. A. Harshman, "Enhanced Line Search: A Novel Method to Accelerate PARAFAC," *SIAM J. Matrix Anal. Appl., Tensor Decompositions and Applications*, 2007, to appear.

[22] D. Nion and L. De Lathauwer, "An Enhanced Line Search Scheme for Complex-Valued Tensor Decompositions. Application in DS-CDMA," *Signal Proc.*, vol. 88, no. 3, pp. 749–755, 2008.

[23] G. Tomasi and R. Bro, "A Comparison of Algorithms for Fitting the PARAFAC Model," *Comp. Stat. Data Anal.*, vol. 50, pp. 1700–1734, 2006.

[24] D. Nion and L. De Lathauwer, "Levenberg-Marquardt computation of the Block Factor Model for Blind Multi-User Access in Wireless Communications," in *Proc. 14th European Signal Processing Conference (Eusipco)*, Florence, IT., 2006.

[25] L. De Lathauwer, B. De Moor, and J. Vandewalle, "A Multilinear Singular Value Decomposition," *SIAM J. Matrix Anal. Appl.*, vol. 21, no. 4, pp. 1253–1278, 2000.

[26] L. De Lathauwer and J. Vandewalle, "Dimensionality Reduction in Higher-Order Signal Processing and Rank-$(r_1, r_2, \ldots, r_n)$ Reduction in Multilinear Algebra," *Lin. Alg. Appl.,*Special Issue on *Linear Algebra in Signal and Image Processing*, vol. 391, pp. 31–55, November 2004.

[27] L. De Lathauwer, B. De Moor, and J. Vandewalle, "On the Best Rank-1 and Rank-$(r_1, r_2, \ldots, r_n)$ Approximation of Higher-Order Tensors," *SIAM J. Matrix Anal. Appl.*, vol. 21, no. 4, pp. 1324–1342, April 2000.

[28] L. De Lathauwer, "A Link between the Canonical Decomposition in Multilinear Algebra and Simultaneous Matrix Diagonalization," *SIAM J. Matrix Anal. Appl.*, vol. 28, no. 3, pp. 642–666, 2006.

[29] L. De Lathauwer and J. Castaing, "Tensor-Based Techniques for the Blind Separation of DS-CDMA signals," *Signal Processing*, Special Issue *Tensor Signal Processing*, vol. 87, no. 2, pp. 322–336, Feb. 2007.

[30] J. B. Kruskal, "Three-way Arrays: Rank and Uniqueness of Trilinear Decompositions, with Application to Arithmetic Complexity and Statistics," *Linear Algebra Appl.*, vol. 18, pp. 95–138, 1977.