

A LINK BETWEEN THE DECOMPOSITION OF A THIRD-ORDER TENSOR IN RANK- $(L,L,1)$ TERMS AND JOINT BLOCK DIAGONALIZATION

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ABSTRACT

In this paper, we show that the Block Component Decomposition in rank- $(L,L,1)$ terms of a third-order tensor, referred to as BCD- $(L,L,1)$, can be reformulated as a Joint Block Diagonalization (JBD) problem, provided that certain assumptions on the dimensions are satisfied. This JBD-based reformulation leads to a new uniqueness bound for the BCD- $(L,L,1)$. We also propose a closed-form solution to solve exact JBD problems. For approximate JBD problems, this closed-form solution yields a good starting value for iterative optimization algorithms. The performance of our technique is illustrated by its application to blind CDMA signal separation.

I. INTRODUCTION

Let us consider the exact Block Component Decomposition (BCD) in rank- $(L,L,1)$ terms of a third-order tensor $\mathcal{T} \in \mathbb{C}^{I \times J \times K}$, introduced in [1]:

$$\mathcal{T} = \sum_{r=1}^R (\mathbf{A}_r \cdot \mathbf{B}_r^T) \circ \mathbf{c}_r, \quad (1)$$

where $\mathbf{A}_r \in \mathbb{C}^{I \times L}$ and $\mathbf{B}_r \in \mathbb{C}^{J \times L}$ are rank- L , $\mathbf{c}_r \in \mathbb{C}^K$, $\forall r \in \{1, \dots, R\}$, \circ is the outer product. The element-wise version of (1) is

$$t_{ijk} = \sum_{r=1}^R c_{kr} \sum_{l=1}^L [\mathbf{A}_r]_{il} [\mathbf{B}_r]_{jl}. \quad (2)$$

The well-known PARAFAC decomposition (see [2]–[5] and references therein) corresponds to the special case of the BCD- $(L,L,1)$ where $L = 1$. Define the partitioned matrices $\mathbf{A} = [\mathbf{A}_1 \dots \mathbf{A}_R] \in \mathbb{C}^{I \times RL}$, $\mathbf{B} = [\mathbf{B}_1 \dots \mathbf{B}_R] \in \mathbb{C}^{J \times RL}$, $\mathbf{C} = [\mathbf{c}_1 \dots \mathbf{c}_R] \in \mathbb{C}^{K \times R}$ and denote by $\mathbf{T}_{IK \times J}$ the $IK \times J$ matrix representation of \mathcal{T} , such that $[\mathbf{T}_{IK \times J}]_{(i-1)K+k,j} = t_{ijk}$. The BCD- $(L,L,1)$ can equivalently be written as

$$\mathbf{T}_{IK \times J} = (\mathbf{A} \odot \mathbf{C}) \cdot \mathbf{B}^T, \quad (3)$$

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where \odot is the Khatri-Rao product, i.e., $\mathbf{A} \odot \mathbf{C} = [\mathbf{A}_1 \otimes \mathbf{c}_1, \dots, \mathbf{A}_R \otimes \mathbf{c}_R]$, where \otimes is the Kronecker product. Computation of the BCD- $(L,L,1)$ of \mathcal{T} consists of the estimation of the unknown matrices \mathbf{A} , \mathbf{B} and \mathbf{C} , which can be done by an Alternating Least Squares (ALS) algorithm [6], possibly coupled with Enhanced Line Search (ELS) to speed up convergence [7], [8].

It is clear that in (1), one can arbitrarily permute the R different rank- $(L,L,1)$ tensors. Also, one can postmultiply \mathbf{A}_r by any nonsingular $(L \times L)$ matrix \mathbf{F}_r , provided that \mathbf{B}_r^T is premultiplied by \mathbf{F}_r^{-1} . Moreover, the factors of a same rank- $(L,L,1)$ term may be arbitrarily scaled, as long as their product remains the same. We call the decomposition essentially unique when it is only subject to these indeterminacies. In [1], sufficient generic conditions for which essential uniqueness of the BCD- $(L,L,1)$ is guaranteed have been derived. We call a property generic when it holds everywhere, except for a set of Lebesgue measure 0.

In [9], we studied the PARAFAC decomposition under the constraint $\min(IJ, K) \leq R$, where the roles of I , J and K can of course be interchanged. We showed that under some conditions, its computation can be reformulated as a Joint Diagonalization (JD) problem. The derivation yields a new uniqueness bound that is more relaxed than the well-known Kruskal bound [10], if the working conditions are satisfied.

In the same spirit as [9], the main motivation of this paper is to work towards a reformulation of the BCD- $(L,L,1)$ in terms of joint diagonalization, and investigate the uniqueness conditions resulting from this reformulation. In [11], we have worked under the conditions $\min(IJ, K) \geq R$ and we have shown that in this case, the computation of the BCD- $(L,L,1)$ can, under some conditions, be reformulated as a JD problem, which yields a uniqueness bound that is more relaxed than the one in [1]. In this paper, we suppose that the long dimension of \mathcal{T} is J (or I , since the roles of \mathbf{A} and \mathbf{B} can be interchanged), rather than K . Our assumptions are:

- (A1) $LR \leq J$ and \mathbf{B} has full column rank,
- (A2) $LR \leq IK$ and $\mathbf{A} \odot \mathbf{C}$ has full column rank.

Under the condition (A1), it has been shown in [1] that essential uniqueness is generically guaranteed if

$$LR \leq J \quad \text{and} \quad \min\left(\left\lfloor \frac{I}{L} \right\rfloor, R\right) + \min(K, R) \geq R + 2. \quad (4)$$

Under (A1) and (A2), the main contributions of this paper are the following:

- i) we show that the computation of the BCD- $(L,L,1)$ can be reformulated as a Joint Block Diagonalization (JBD) problem, even in the underdetermined case where $LR > I$;
- ii) we propose a closed-form solution for the exact JBD problem. For noisy JBD problems, this solution can be used as a good starting point for existing optimization algorithms;
- iii) the JBD reformulation leads to a new uniqueness bound that is more relaxed than (4);
- iv) we illustrate our results by means of an application in CDMA.

II. LINK BETWEEN THE BCD- $(L,L,1)$ AND JOINT BLOCK DIAGONALIZATION

Under assumptions (A1) and (A2), it is clear from (3) that $\mathbf{T}_{IK \times J}$ is of rank LR . Let us write the economy-size SVD of $\mathbf{T}_{IK \times J}$:

$$\mathbf{T}_{IK \times J} = \mathbf{U} \cdot \mathbf{\Sigma} \cdot \mathbf{V}^H = \mathbf{E} \cdot \mathbf{V}^H, \quad (5)$$

where $\mathbf{U} \in \mathbb{C}^{IK \times LR}$ and $\mathbf{V} \in \mathbb{C}^{J \times LR}$ are column-wise orthonormal and $\mathbf{\Sigma} \in \mathbb{C}^{LR \times LR}$ is diagonal. Combination of (3) and (5) yields:

$$\mathbf{E} = (\mathbf{A} \odot \mathbf{C}) \cdot \mathbf{W}^{-1}, \quad (6)$$

$$\mathbf{B}^T = \mathbf{W}^{-1} \cdot \mathbf{V}^H, \quad (7)$$

in which $\mathbf{W} \in \mathbb{C}^{LR \times LR}$ is an a priori unknown nonsingular matrix. From the indeterminacies of the BCD- $(L,L,1)$, it follows that \mathbf{W} can only be determined up to right multiplication with a block diagonal matrix involving nonsingular $L \times L$ blocks. If we obtain an estimate of \mathbf{W} , an estimate of \mathbf{B} follows from (7) and an estimate of $\mathbf{A} \odot \mathbf{C}$ follows from (6). To obtain \mathbf{A} and \mathbf{C} from $\mathbf{A} \odot \mathbf{C}$, we proceed as follows. We first compute $\mathbf{E} \cdot \mathbf{W} = [\mathbf{F}_1, \dots, \mathbf{F}_R]$, where $\mathbf{F}_r \in \mathbb{C}^{IK \times L}$, $1 \leq r \leq R$. Since \mathbf{F}_r is an estimate of $\mathbf{A}_r \otimes \mathbf{c}_r$, we first stack \mathbf{F}_r in $\bar{\mathbf{F}}_r \in \mathbb{C}^{K \times IL}$. Then, the best rank-1 approximation of $\bar{\mathbf{F}}_r$ yields an estimate of \mathbf{c}_r and $\text{vec}(\mathbf{A}_r^T)$, $1 \leq r \leq R$. Here, $\text{vec}(\mathbf{X}) = [\mathbf{x}_1^T, \mathbf{x}_2^T, \dots, \mathbf{x}_N^T]^T$ is the vector in which all the columns of $\mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N]$ are stacked.

The problem is now the estimation of the nonsingular matrix \mathbf{W} that links the matrix factorizations (6) and (7). In the following, we show that this matrix can be obtained by solving a JBD problem. We first need the following tool for rank-1 detection.

Theorem 1. Consider the bilinear mapping

$\Phi : (\mathbf{X}, \mathbf{Y}) \in \mathbb{C}^{I \times J} \times \mathbb{C}^{I \times J} \mapsto \Phi(\mathbf{X}, \mathbf{Y}) \in \mathbb{C}^{I \times I \times J \times J}$ defined by:

$$\begin{aligned} (\Phi(\mathbf{X}, \mathbf{Y}))_{i_1 i_2 j_1 j_2} &= x_{i_1 j_1} y_{i_2 j_2} + y_{i_1 j_1} x_{i_2 j_2} \\ &\quad - x_{i_1 j_2} y_{i_2 j_1} - y_{i_1 j_2} x_{i_2 j_1}. \end{aligned}$$

Then, we have $\Phi(\mathbf{X}, \mathbf{X}) = \mathcal{O}$ if and only if \mathbf{X} is at most rank-1 [9], [12], where \mathcal{O} is the zero tensor. ■

We consider the partitioning $\mathbf{E} = [\mathbf{E}_1, \dots, \mathbf{E}_R]$, in which $\mathbf{E}_r \in \mathbb{C}^{IK \times L}$. Let us compute, for all $r, s \in [1, LR]$:

$$\begin{aligned} \mathcal{P}_{rs} &= \Phi(\mathbf{E}_r, \mathbf{E}_s) \\ &= \sum_{u,v=1}^{LR} (\mathbf{W}^{-1})_{ur} (\mathbf{W}^{-1})_{vs} \Phi([\mathbf{A} \odot \mathbf{C}]_{:,u}, [\mathbf{A} \odot \mathbf{C}]_{:,v}), \end{aligned} \quad (8)$$

due to the bilinearity of Φ . We have used the MATLAB notation $[\mathbf{X}]_{:,u}$ to denote the u th column of \mathbf{X} . We introduce new indices $\alpha, \beta \in [1, R]$ and $l, m \in [1, L]$ such that $u = (\alpha - 1)L + l$ and $v = (\beta - 1)L + m$. Then we have:

$$\begin{aligned} \mathcal{P}_{rs} &= \sum_{l,m=1}^L \sum_{\alpha,\beta=1}^R (\mathbf{W}^{-1})_{(\alpha-1)L+l,r} (\mathbf{W}^{-1})_{(\beta-1)L+m,s} \\ &\quad \Phi([\mathbf{A}_\alpha]_{:,l} \mathbf{c}_\alpha^T, [\mathbf{A}_\beta]_{:,m} \mathbf{c}_\beta^T). \end{aligned} \quad (9)$$

Now assume that there exists a symmetric matrix $\mathbf{M} \in \mathbb{C}^{LR \times LR}$ such that

$$\sum_{r,s=1}^{LR} m_{rs} \mathcal{P}_{rs} = \mathcal{O}, \quad (10)$$

(we will motivate this assumption below). Substitution of (9) in (10) yields:

$$\sum_{r,s=1}^{LR} \sum_{l,m=1}^L \sum_{\alpha,\beta=1}^R (\mathbf{W}^{-1})_{(\alpha-1)L+l,r} (\mathbf{W}^{-1})_{(\beta-1)L+m,s} \quad (11)$$

$$m_{rs} \Phi([\mathbf{A}_\alpha]_{:,l} \mathbf{c}_\alpha^T, [\mathbf{A}_\beta]_{:,m} \mathbf{c}_\beta^T) = \mathcal{O}$$

From the definition of the mapping Φ , we have that $\Phi([\mathbf{A}_\alpha]_{:,l} \mathbf{c}_\alpha^T, [\mathbf{A}_\beta]_{:,m} \mathbf{c}_\beta^T) = \mathcal{O}$ if $\alpha = \beta$, regardless of the values of l and m . We now make the crucial assumption that the $L^2 C_R^2 = \frac{RL^2(R-1)}{2}$ tensors of the set

$$\begin{aligned} \Omega &= \{ \Phi([\mathbf{A}_\alpha]_{:,l} \mathbf{c}_\alpha^T, [\mathbf{A}_\beta]_{:,m} \mathbf{c}_\beta^T), \\ &\quad 1 \leq \alpha < \beta \leq R, 1 \leq l, m \leq L \}, \end{aligned} \quad (12)$$

are linearly independent. It follows that (11) is equivalent to

$$\mathbf{W}^{-1} \cdot \mathbf{M} \cdot \mathbf{W}^{-T} = \mathbf{D}, \quad (13)$$

in which \mathbf{D} is an $LR \times LR$ block diagonal matrix with symmetric $L \times L$ blocks on its diagonal. Equivalently,

$$\mathbf{M} = \mathbf{W} \cdot \mathbf{D} \cdot \mathbf{W}^T. \quad (14)$$

It can be verified that any matrix \mathbf{M} that has the structure (14), satisfies (10). Hence, there are $N = \frac{RL(L+1)}{2}$ symmetric matrices in the kernel of (10), which can all be decomposed as in (14):

$$\mathbf{M}_n = \mathbf{W} \cdot \mathbf{D}_n \cdot \mathbf{W}^T, \quad n = 1, \dots, N. \quad (15)$$

Finally, an estimate of \mathbf{W} is obtained by solving the JBD problem (15). This can be done by existing algorithms for non-unitary JBD, e.g., [13]. Alternatively, this can also be done by the computation of the BCD- $(L,L,1)$ [1] of the tensor $\mathcal{M} \in \mathbb{C}^{LR \times LR \times N}$ obtained by stacking the matrices \mathbf{M}_n . The latter decomposition can be computed by means of the

algorithms proposed in [6], [8], [14]. In this paper, we will solve the JBD problem via the computation of the BCD- (L, L, \cdot) by means of an Alternating Least Squares algorithm with Enhanced Line Search (ALS+ELS) [7], [8]. In the next section, we propose a closed-form solution of the exact JBD problem (15).

III. A CLOSED-FORM SOLUTION TO THE EXACT JBD PROBLEM

Assume that $N \geq 3$ and consider the partitioning $\mathbf{W} = [\mathbf{W}_1, \dots, \mathbf{W}_R]$, with $\mathbf{W}_r \in \mathbb{C}^{RL \times L}$, $r = 1, \dots, R$. Assume that (15) is exactly satisfied. For simplicity, we assume that \mathbf{M}_2 is full rank. Then, we have:

$$\mathbf{M}_1 \cdot \mathbf{M}_2^{-1} = \mathbf{W} \cdot (\mathbf{D}_1 \cdot \mathbf{D}_2^{-1}) \cdot \mathbf{W}^{-1}. \quad (16)$$

This implies that the column space of \mathbf{W}_r , $r = 1, \dots, R$, is an invariant subspace of $\mathbf{M}_1 \cdot \mathbf{M}_2^{-1}$. We compute the Eigenvalue Decomposition (EVD) of $\mathbf{M}_1 \cdot \mathbf{M}_2^{-1}$:

$$\mathbf{M}_1 \cdot \mathbf{M}_2^{-1} = \mathbf{E} \cdot \mathbf{\Lambda} \cdot \mathbf{E}^{-1} \quad (17)$$

(or the generalized EVD of $(\mathbf{M}_1, \mathbf{M}_2)$). From (17), it is unclear which eigenvectors have to be paired. Taking the problem indeterminacies into account, we have that \mathbf{W} can be written as

$$\mathbf{W} = \mathbf{E} \cdot \mathbf{\Pi}, \quad (18)$$

where $\mathbf{\Pi}$ is an a priori unknown permutation matrix that pairs the eigenvectors L by L . Estimation of $\mathbf{\Pi}$ can be achieved by checking the permuted block-diagonal structure of the matrices $\mathbf{E}^{-1} \cdot \mathbf{M}_n \cdot \mathbf{E}$, $3 \leq n \leq N$.

IV. UNIQUENESS OF THE BCD- $(L, L, 1)$

Together with (A1) and (A2), the independence of the set Ω forms a set of deterministic conditions under which essential uniqueness of BCD- $(L, L, 1)$ is guaranteed. Moreover, under these conditions, the solution can be found by JBD and the noise-free solution is known explicitly. We now present a generic formulation of these conditions. Generically, \mathbf{B} has full column rank if $LR \leq J$. It can be shown that, generically, $\mathbf{A} \odot \mathbf{C}$ has full column rank if $LR \leq IK$. The definition of the mapping Φ implies that the tensors in Ω have only $C_I^2 C_K^2 = (1/4)I(I-1)K(K-1)$ distinct non-zero entries. Hence, the tensors in Ω can be stacked in an $(C_I^2 C_K^2 \times L^2 C_R^2)$ matrix \mathbf{P} and the question is whether \mathbf{P} has rank $L^2 C_R^2$. We conjecture that this is generically the case as long as \mathbf{P} has at least as many rows as columns. Hence, we have the following sufficient condition for generic uniqueness.

Conjecture. The BCD- $(L, L, 1)$ is generically essentially unique if

$$LR \leq \min(IK, J) \text{ and } C_I^2 C_K^2 \geq L^2 C_R^2. \quad (19)$$

Note that for $L = 1$, i.e., for PARAFAC, this bound reduces to the bound derived in [9]. Condition (19) is significantly more relaxed than (4) if $LR \leq \min(IK, J)$.

According to [15, Theorem 5.A.2], a set is generically linearly independent if it is independent for one set of parameters. Hence, for given values of I, J, K and L , the conjecture can always be proved by checking the rank of $\mathbf{A} \odot \mathbf{C}$ and \mathbf{P} for one random choice of \mathbf{A}, \mathbf{B} and \mathbf{C} .

V. SIMULATION RESULTS

In this section, we apply our method to the blind separation of multi-user DS-CDMA signals. The channel is convolutive but the multipath reflections occur only in the far field of the receive antenna array. The parameters are the following: I is the spreading factor, J the number of transmitted symbols, K the number of receive antennas, R the number of users and L the duration (in number of symbol periods) of the channel impulse response of each user. It has been shown in [11], [16], [17] that this signal can be seen as an $I \times J \times K$ tensor \mathcal{T} that admits a BCD- $(L, L, 1)$, up to noise. Here, the $J \times L$ matrix \mathbf{B}_r holds the J symbols of user r and has a Toeplitz structure. The channel impulse response coefficients are stacked in the $I \times L$ matrix \mathbf{A}_r . In this application, the second dimension J is naturally the ‘‘long’’ dimension. We compare the performance of the following algorithms:

M1. BCD- $(L, L, 1)$ computed by ALS+ELS [8] with one random initialization, where the Toeplitz structure of the matrices \mathbf{B}_r is enforced at each iteration (by updating the Toeplitz generator vectors directly, as explained in [14]);

M2. BCD- $(L, L, 1)$ computed by ALS+ELS [8] with one random initialization, where the Toeplitz structure of the matrices \mathbf{B}_r is only recovered after convergence. The latter is done by means of the subspace method in [18];

M3. BCD- $(L, L, 1)$ computed by JBD, initialized with the closed-form solution, followed by M1 to refine the estimate;

M4. BCD- $(L, L, 1)$ computed by JBD, initialized with the closed-form solution, followed by M2 to refine the estimate.

We have conducted experiments consisting of 200 Monte-Carlo runs for each value of the Signal to Noise Ratio (SNR), the latter varying from 0 dB to 40 dB. We have $R = 5$, $L = 2$, $K = 4$, $I = 8$ and $J = 30$ BPSK symbols. In each run, the matrices \mathbf{A} and \mathbf{C} are redrawn from a zero-mean unit-variance i.i.d. Gaussian distribution. In this setting, the conditions in [17] are not satisfied. The latter generically amount to $\min(I, J) \geq LR$ and $K \geq 2$.

Let \mathcal{T} be the observed tensor and $\hat{\mathcal{T}}$ the tensor constructed using the estimates of \mathbf{A}, \mathbf{B} and \mathbf{C} . In Fig. 1(a), we plot the evolution of the average norm of the residual tensor $\|\mathcal{T} - \hat{\mathcal{T}}\|_F$ for M3, after the JBD stage and after the ALS+ELS refinement stage. For low SNR, the solution obtained from the JBD stage is still far from the optimal least squares solution, and the ALS+ELS final refinement stage significantly improves the global fit. For high SNR, the JBD solution is as good as the optimal least squares solution.

In Fig. 1(b), we plot the evolution of the Mean Squared Error (MSE) of the symbol matrix. We also compare to the

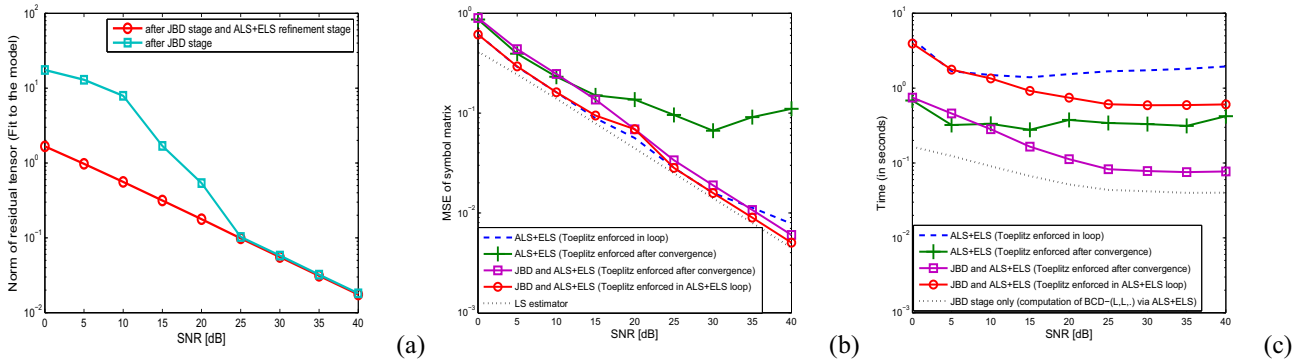


Fig. 1. Performance of M1, M2, M3, M4 for varying SNR. (a) Evolution of Fit. (b) Evolution of MSE of symbol matrix. (c) Evolution of execution time.

Least Squares (LS) estimator, in which perfect knowledge of the antenna response matrix \mathbf{C} and the channel matrix \mathbf{A} is assumed. We observe that M1 and M3 perform similarly and that they are close to the LS estimator. This good performance results from the Toeplitz structure preservation strategy inside the ALS+ELS steps. When this structure is not preserved but only recovered after convergence, the MSE is worse (M2 and M4). However, the advantage of a good initialization is clear.

In Fig. 1(c), we compare the evolution of the total execution times. We observe that M3 (resp. M4) converges faster than M1 (resp. M2), which emphasizes again the importance of the starting value. This is also clear from the number of iterations (not shown here). The JBD stage has a low cost compared to the refinement stage. Imposing the Toeplitz structure in the loop is more accurate but also more expensive than imposing the structure after convergence.

VI. CONCLUSION

In this paper, we have established a link between the BCD- $(L, L, 1)$ and Joint Block Diagonalization (JBD). For the exact problem, a closed-form solution has been obtained, even in the underdetermined case. A deterministic sufficient condition for essential uniqueness has been derived. An easy-to-check generic version of this condition was formulated. The new condition is significantly more relaxed than the existing conditions. In this paper, we have for simplicity assumed that all terms have the same multilinear rank $(L, L, 1)$. In the journal version, we will allow the value of L to depend on the value of r . Although the solution obtained from JBD is still relatively far from the optimal least squares solution for high noise levels, it can be regarded as a relatively cheap way to initialize existing algorithms.

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