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DECOMPOSITIONS OF A HIGHER-ORDER TENSOR IN BLOCK TERMS—PART I: LEMMAS FOR PARTITIONED MATRICES*

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Abstract. In this paper we study a generalization of Kruskal's permutation lemma to partitioned matrices. We define the k'-rank of partitioned matrices as a generalization of the k-rank of matrices. We derive a lower-bound on the k'-rank of Khatri–Rao products of partitioned matrices. We prove that Khatri–Rao products of partitioned matrices are generically full column rank.

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1. Introduction.

1.1. Organization of the paper. In a companion paper we introduce decompositions of a higher-order tensor in several types of block terms [3]. For the analysis of these decompositions, we need a number of tools. Some of these are introduced in the present paper. In section 2 we derive a generalization of Kruskal's permutation lemma [6], which we call the equivalence lemma for partitioned matrices. Section 2 also introduces the k'-rank of partitioned matrices as a generalization of the k-rank of matrices [6]. In section 3 we present some results on the rank and k'-rank of Khatri–Rao products of partitioned matrices (see (1.1)).

1.2. Notation. We use $\mathbb{K}$ to denote $\mathbb{R}$ or $\mathbb{C}$ when the difference is not important. In this paper scalars are denoted by lowercase letters (a, b, ...), vectors are written in boldface lowercase ($\mathbf{a}$, $\mathbf{b}$, ...), and matrices correspond to boldface capitals ($\mathbf{A}$, $\mathbf{B}$, ...). This notation is consistently used for lower-order parts of a given structure. For instance, the entry with row index i and column index j in a matrix $\mathbf{A}$, i.e., $(\mathbf{A})_{ij}$, is symbolized by a_{ij} (also $(\mathbf{a})_i = a_i$). If no confusion is possible, the ith column vector of a matrix $\mathbf{A}$ is denoted as $\mathbf{a}_i$, i.e., $\mathbf{A} = [\mathbf{a}_1\ \mathbf{a}_2\ \ldots]$. Sometimes we use the MATLAB colon notation to indicate submatrices of a given matrix or subtensors of a given tensor. Italic capitals are also used to denote index upper bounds (e.g., $i = 1, 2, \ldots, I$). The symbol $\otimes$ denotes the Kronecker product,

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \ldots \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \ldots \\ \vdots & \vdots & \end{pmatrix}.$$

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Let $\mathbf{A} = [\mathbf{A}_1 \ \dots \ \mathbf{A}_R]$ and $\mathbf{B} = [\mathbf{B}_1 \ \dots \ \mathbf{B}_R]$ be two partitioned matrices. Then the Khatri–Rao product is defined as the partitionwise Kronecker product and represented by $\odot$ [7]:

$$\mathbf{A} \odot \mathbf{B} = (\mathbf{A}_1 \otimes \mathbf{B}_1 \dots \mathbf{A}_R \otimes \mathbf{B}_R). \tag{1.1}$$

In recent years, the term "Khatri–Rao product" and the symbol $\odot$ have been used mainly in cases where $\mathbf{A}$ and $\mathbf{B}$ are partitioned into vectors. For clarity, we denote this particular, columnwise Khatri–Rao product by $\odot_c$:

$$\mathbf{A} \odot_c \mathbf{B} = (\mathbf{a}_1 \otimes \mathbf{b}_1 \dots \mathbf{a}_R \otimes \mathbf{b}_R).$$

The column space of a matrix and its orthogonal complement will be denoted by $\mathrm{span}(\mathbf{A})$ and $\mathrm{null}(\mathbf{A})$. The rank of a matrix $\mathbf{A}$ will be denoted by $\mathrm{rank}(\mathbf{A})$ or $r_{\mathbf{A}}$. The superscripts $\cdot^T$, $\cdot^H$, and $\cdot^\dagger$ denote the transpose, complex conjugated transpose, and Moore–Penrose pseudoinverse, respectively. The $(N \times N)$ identity matrix is represented by $\mathbf{I}_{N\times N}$. The $(I \times J)$ zero matrix is denoted by $\mathbf{0}_{I\times J}$.

2. The equivalence lemma for partitioned matrices. Let $\omega(\mathbf{x})$ denote the number of nonzero entries of a vector $\mathbf{x}$. The following lemma was originally proposed by Kruskal in [6]. It is known as the *permutation lemma*. It plays a crucial role in the analysis of the uniqueness of the canonical/parallel factor (CANDECOMP/PARAFAC) decomposition [1, 5]. The proof was reformulated in terms of accessible basic linear algebra in [9]. An alternative proof was given in [4]. The link between the two proofs is also discussed in [9].

LEMMA 2.1 (permutation lemma). *Consider two matrices $\bar{\mathbf{A}}, \mathbf{A} \in \mathbb{K}^{I\times R}$ that have no zero columns. If for every vector $\mathbf{x}$ such that $\omega(\mathbf{x}^T\bar{\mathbf{A}}) \leqslant R - r_{\bar{\mathbf{A}}} + 1$, we have $\omega(\mathbf{x}^T\mathbf{A}) \leqslant \omega(\mathbf{x}^T\bar{\mathbf{A}})$, then there exists a unique permutation matrix $\mathbf{\Pi}$ and a unique nonsingular diagonal matrix $\mathbf{\Lambda}$ such that $\bar{\mathbf{A}} = \mathbf{A} \cdot \mathbf{\Pi} \cdot \mathbf{\Lambda}$.*

Below, we present a generalization of the permutation lemma for matrices that are partitioned as in $\mathbf{A} = [\mathbf{A}_1 \dots \mathbf{A}_R]$. This generalization is essential in the study of the uniqueness of the decompositions introduced in [3].

Let us first introduce some additional prerequisites. Let $\omega'(\mathbf{x})$ denote the number of parts of a partitioned vector $\mathbf{x}$ that are not all-zero. We call the partitioning of a partitioned matrix $\mathbf{A}$ uniform when all submatrices are of the same size. We also have the following definition.

DEFINITION 2.2. *The* Kruskal rank *or* k-rank *of a matrix $\mathbf{A}$, denoted by $\mathrm{rank}_k(\mathbf{A})$ or $k_{\mathbf{A}}$, is the maximal number r such that any set of r columns of $\mathbf{A}$ is linearly independent* [6].

We call a property generic when it holds with probability one when the parameters of the problem are drawn from continuous probability density functions. Let $\mathbf{A} \in \mathbb{K}^{I\times R}$. Generically, we have $k_{\mathbf{A}} = \min(I, R)$. K-ranks appear in the formulation of the famous Kruskal condition for CANDECOMP/PARAFAC uniqueness (see [3, Theorem 1.14]).

We now generalize the k-rank concept to partitioned matrices.

DEFINITION 2.3. *The* k'-rank *of a (not necessarily uniformly) partitioned matrix $\mathbf{A}$, denoted by $\mathrm{rank}_{k'}(\mathbf{A})$ or $k'_{\mathbf{A}}$, is the maximal number r such that any set of r submatrices of $\mathbf{A}$ yields a set of linearly independent columns.*

Let $\mathbf{A} \in \mathbb{K}^{I\times LR}$ be uniformly partitioned in R matrices $\mathbf{A}_r \in \mathbb{K}^{I\times L}$. Generically, we have $k'_{\mathbf{A}} = \min(\lfloor \frac{I}{L} \rfloor, R)$. K'-ranks will appear in the formulation of generalizations of Kruskal's condition to block term decompositions [3].

The generalization of the permutation lemma to partitioned matrices is now as follows.

LEMMA 2.4 (equivalence lemma for partitioned matrices). *Consider* $\bar{\mathbf{A}}, \mathbf{A} \in \mathbb{K}^{I \times \sum_{r=1}^{R} L_r}$, *partitioned in the same but not necessarily uniform way into* R *submatrices that are full column rank. Suppose that for every* $\mu \leqslant R - k'_{\bar{\mathbf{A}}} + 1$ *there holds that for a generic*[1] *vector* $\mathbf{x}$ *such that* $\omega'(\mathbf{x}^H \bar{\mathbf{A}}) \leqslant \mu$, *we have* $\omega'(\mathbf{x}^H \mathbf{A}) \leqslant \omega'(\mathbf{x}^H \bar{\mathbf{A}})$. *Then there exists a unique block-permutation matrix* $\mathbf{\Pi}$ *and a unique nonsingular block-diagonal matrix* $\mathbf{\Lambda}$, *such that* $\bar{\mathbf{A}} = \mathbf{A} \cdot \mathbf{\Pi} \cdot \mathbf{\Lambda}$, *where the block-transformation is compatible with the block-structure of* $\mathbf{A}$ *and* $\bar{\mathbf{A}}$.

The permutation lemma is not only about permutations. Rather it gives a condition under which two matrices are *equivalent* up to columnwise permutation *and scaling*. The lemma thus makes sure that two matrices belong to the same quotient class of the equivalence relation defined by $\mathbf{A} \sim \mathbf{B} \Leftrightarrow \mathbf{A} = \mathbf{B} \cdot \mathbf{\Pi} \cdot \mathbf{\Lambda}$, in which $\mathbf{\Pi}$ is an arbitrary permutation matrix and $\mathbf{\Lambda}$ an arbitrary nonsingular diagonal matrix, respectively. We find it therefore appropriate to call Lemma 2.4 the *equivalence* lemma for partitioned matrices.

We note that the rank $r_{\bar{\mathbf{A}}}$ in the permutation lemma has been replaced by the k'-rank $k'_{\bar{\mathbf{A}}}$ in Lemma 2.4, because the permutation lemma admits a simpler proof when we can assume that $r_{\bar{\mathbf{A}}} = k_{\bar{\mathbf{A}}}$. It is this simpler proof, given in [4], that will be generalized in this paper. We stay quite close to the text of [4]. We recommend studying the proof in [4] before reading the remainder of this section.

We work as follows. First we have a closer look at the meaning of the condition in the equivalence lemma for partitioned matrices (Lemma 2.5). Then we prove that $\mathbf{A}$ and $\bar{\mathbf{A}}$ are equivalent when the condition in the equivalence lemma for partitioned matrices holds for all $\mu \leqslant R$ (Lemma 2.6). Finally we show that it is sufficient to claim that the condition holds for $\mu \leqslant R - k'_{\bar{\mathbf{A}}} + 1$ (Lemma 2.7).

LEMMA 2.5. *Consider* $\bar{\mathbf{A}}, \mathbf{A} \in \mathbb{K}^{I \times L}$, *partitioned in the same but not necessarily uniform way into* R *submatrices that are full column rank. The following two statements are equivalent:*

(i) *For every* $\mu \leqslant R - k'_{\bar{\mathbf{A}}} + 1$ *there holds that for a generic vector* $\mathbf{x}$ *such that* $\omega'(\mathbf{x}^H \bar{\mathbf{A}}) \leqslant \mu$, *we have* $\omega'(\mathbf{x}^H \mathbf{A}) \leqslant \omega'(\mathbf{x}^H \bar{\mathbf{A}})$.

(ii) *If a vector is orthogonal to* $c \geqslant k'_{\bar{\mathbf{A}}} - 1$ *submatrices of* $\bar{\mathbf{A}}$, *then it must generically be orthogonal to at least* c *submatrices of* $\mathbf{A}$.

These, in turn, imply the following:

(iii) *For every set of* $c \geqslant k'_{\bar{\mathbf{A}}} - 1$ *submatrices of* $\bar{\mathbf{A}}$, *there exists a set of at least* c *submatrices of* $\mathbf{A}$ *such that span(matrix formed by these* $c \geqslant k'_{\bar{\mathbf{A}}} - 1$ *submatrices of* $\bar{\mathbf{A}}$) $\supseteq$ *span(matrix formed by the* c *or more submatrices of* $\mathbf{A}$).

Proof. The equivalence of (i) and (ii) follows directly from the definition of $\omega'(\mathbf{x})$.

[1] We mean the following. Consider, for instance, a partitioned matrix $\bar{\mathbf{A}} = [\mathbf{a}_1\ \mathbf{a}_2 | \mathbf{a}_3\ \mathbf{a}_4] \in \mathbb{K}^{4 \times 4}$ that is full column rank. The set $\mathsf{S} = \{\mathbf{x} | \omega'(\mathbf{x}^H \bar{\mathbf{A}}) \leqslant 1\}$ is the union of two subspaces, S_1 and S_2, consisting of the set of vectors orthogonal to $\{\mathbf{a}_1, \mathbf{a}_2\}$ and $\{\mathbf{a}_3, \mathbf{a}_4\}$, respectively. When we say that for a generic vector $\mathbf{x}$ such that $\omega'(\mathbf{x}^H \bar{\mathbf{A}}) \leqslant 1$, we have $\omega'(\mathbf{x}^H \mathbf{A}) \leqslant \omega'(\mathbf{x}^H \bar{\mathbf{A}})$, we mean that $\omega'(\mathbf{x}^H \mathbf{A}) \leqslant \omega'(\mathbf{x}^H \bar{\mathbf{A}})$ holds with probability one for a vector $\mathbf{x}$ drawn from a continuous probability density function over S_1 and that $\omega'(\mathbf{x}^H \mathbf{A}) \leqslant \omega'(\mathbf{x}^T \bar{\mathbf{A}})$ also holds with probability one for a vector $\mathbf{x}$ drawn from a continuous probability density function over S_2. In general, the set $\mathsf{S} = \{\mathbf{x} | \omega'(\mathbf{x}^H \bar{\mathbf{A}}) \leqslant \mu\}$ consists of a finite union of subspaces, where we count only the subspaces that are not contained in another subspace. For each of these subspaces, the property should hold with probability one for a vector $\mathbf{x}$ drawn from a continuous probability density function over that subspace.

We now prove in two ways that (ii) implies (iii). The first proof is a generalization of [4, Remark 1]. This proof is by contradiction. Suppose that there is a set of $c_0 \geqslant k'_{\bar{\mathbf{A}}} - 1$ submatrices of $\bar{\mathbf{A}}$, say, $\bar{\mathbf{A}}_1, \ldots, \bar{\mathbf{A}}_{c_0}$, and that there are only $c_0 - k$ submatrices of $\mathbf{A}$, say, $\mathbf{A}_1, \ldots, \mathbf{A}_{c_0-k}$, such that

$$\operatorname{span}([\bar{\mathbf{A}}_1 \ \ldots \ \bar{\mathbf{A}}_{c_0}]) \supseteq \operatorname{span}([\mathbf{A}_1 \ \ldots \ \mathbf{A}_{c_0-k}]),$$

where $1 \leqslant k \leqslant c_0$. The column space of none of the remaining submatrices of $\mathbf{A}$, i.e., $\mathbf{A}_{c_0-k+1}, \ldots, \mathbf{A}_R$, is contained in $\operatorname{span}([\bar{\mathbf{A}}_1 \ \ldots \ \bar{\mathbf{A}}_{c_0}])$; otherwise, k can be reduced. This implies that for every $i = c_0 - k + 1, \ldots, R$, there exists a certain nonzero vector $\mathbf{x}_i \in \operatorname{null}([\bar{\mathbf{A}}_1 \ \ldots \ \bar{\mathbf{A}}_{c_0}])$ such that

$$\mathbf{x}_i^H \mathbf{A}_i \neq [0 \ \ldots \ 0]. \tag{2.1}$$

We can assume that $\operatorname{null}([\bar{\mathbf{A}}_1 \ \ldots \ \bar{\mathbf{A}}_{c_0}])$ is a subspace of dimension $m \geqslant 1$. The case $m = 0$ corresponds to $\operatorname{span}([\bar{\mathbf{A}}_1 \ \ldots \ \bar{\mathbf{A}}_{c_0}]) = \mathbb{K}^I$. In this case, the span of all submatrices of $\mathbf{A}$ is contained in $\operatorname{span}([\bar{\mathbf{A}}_1 \ \ldots \ \bar{\mathbf{A}}_{c_0}])$.

Due to the existence of $\mathbf{x}_i$ in (2.1), we have for $i = c_0 - k + 1, \ldots, R$ that $\operatorname{null}([\bar{\mathbf{A}}_1 \ \ldots \ \bar{\mathbf{A}}_{c_0} \mathbf{A}_i])$ is a proper subspace of $\operatorname{null}([\bar{\mathbf{A}}_1 \ \ldots \ \bar{\mathbf{A}}_{c_0}])$ with dimension at most $m-1$. Since the union of a countable number of at most $(m-1)$-dimensional subspaces of $\mathbb{K}^I$ cannot cover an m-dimensional subspace of $\mathbb{K}^I$, there holds for a generic vector $\mathbf{x}_0 \in \operatorname{null}([\bar{\mathbf{A}}_1 \ \ldots \ \bar{\mathbf{A}}_{c_0}])$ that

$$\mathbf{x}_0^H \mathbf{A}_i \neq [0 \ \ldots \ 0], \quad i = c_0 - k + 1, \ldots, R.$$

We have a contradiction with (ii).

The second proof is direct.[2] If a vector is orthogonal to c submatrices of $\bar{\mathbf{A}}$, then it is in the left null space of c submatrices of $\bar{\mathbf{A}}$. Denote the matrix formed by these c submatrices by $\bar{\mathbf{A}}_c$. By assumption, we have that the vector is generically also in the left null space of $\bar{c} \geqslant c$ submatrices of $\mathbf{A}$. Denote the matrix formed by these $\bar{c}$ submatrices by $\mathbf{A}_{\bar{c}}$. Since

$$\operatorname{null}(\bar{\mathbf{A}}_c) \subseteq \operatorname{null}(\mathbf{A}_{\bar{c}})$$

we have

$$\operatorname{span}(\bar{\mathbf{A}}_c) \supseteq \operatorname{span}(\mathbf{A}_{\bar{c}}).$$

This completes the proof. □

We now demonstrate the equivalence of matrices under a condition that seems stronger than the one in the equivalence lemma for partitioned matrices.

LEMMA 2.6. *Consider $\bar{\mathbf{A}}, \mathbf{A} \in \mathbb{K}^{I \times L}$, partitioned in the same but not necessarily uniform way into R submatrices that are full column rank. The following two statements are equivalent:*

(i) *There exists a unique block-permutation matrix $\mathbf{\Pi}$ and a unique nonsingular block-diagonal matrix $\mathbf{\Lambda}$, such that $\bar{\mathbf{A}} = \mathbf{A} \cdot \mathbf{\Pi} \cdot \mathbf{\Lambda}$, where the block-transformation is compatible with the block-structure of $\mathbf{A}$ and $\bar{\mathbf{A}}$.*

(ii) *For every $\mu \leqslant R$ there holds that, for a generic vector $\mathbf{x}$ such that $\omega'(\mathbf{x}^H \bar{\mathbf{A}}) \leqslant \mu$, we have $\omega'(\mathbf{x}^H \mathbf{A}) \leqslant \omega'(\mathbf{x}^H \bar{\mathbf{A}})$.*

[2]This proof was suggested by an anonymous reviewer.

Proof. The implication of (ii) from (i) is trivial. The implication of (i) from (ii) is proved by induction on the number of submatrices R.

For $R = 1$, the condition in the lemma means that $\omega'(\mathbf{x}^H\mathbf{A}) = 0$ for a generic vector $\mathbf{x}$ satisfying $\omega'(\mathbf{x}^H\bar{\mathbf{A}}) = 0$. This implies that $\text{null}(\bar{\mathbf{A}}) \subseteq \text{null}(\mathbf{A})$. Since $\text{null}(\mathbf{A})$ and $\text{null}(\bar{\mathbf{A}})$ are the orthogonal complements of $\text{span}(\mathbf{A})$ and $\text{span}(\bar{\mathbf{A}})$, respectively, we have $\text{span}(\mathbf{A}) \subseteq \text{span}(\bar{\mathbf{A}})$. Since both $\mathbf{A}$ and $\bar{\mathbf{A}}$ are full column rank, the dimensions of $\text{span}(\mathbf{A})$ and $\text{span}(\bar{\mathbf{A}})$ are equal. Hence, we have $\text{span}(\mathbf{A}) = \text{span}(\bar{\mathbf{A}})$ and $\mathbf{A} = \bar{\mathbf{A}} \cdot \mathbf{\Lambda}$, where $\mathbf{\Lambda}$ is $(L \times L)$ nonsingular.

Now assume that the lemma holds for all $R \leqslant K$. We show that it then also holds for $R = K + 1$. The proof is by contradiction. We assume that in the induction step matrices $\mathbf{A}_1$ and $\bar{\mathbf{A}}_1$ are appended to $[\mathbf{A}_2 \ \ldots \ \mathbf{A}_{K+1}]$ and $[\bar{\mathbf{A}}_2 \ \ldots \ \bar{\mathbf{A}}_{K+1}]$, respectively. Both $\mathbf{A}_1$ and $\bar{\mathbf{A}}_1$ have L_1 columns. Without loss of generality, we assume that none of the other submatrices $\mathbf{A}_2, \ldots, \mathbf{A}_{K+1}, \bar{\mathbf{A}}_2, \ldots, \bar{\mathbf{A}}_{K+1}$ has less than L_1 columns.

Assume that $\text{span}(\bar{\mathbf{A}}_1)$ does not coincide with $\text{span}(\mathbf{A}_j)$ for any $j = 1, \ldots, R = K+1$. This means that for all j, $\text{span}([\bar{\mathbf{A}}_1 \ \mathbf{A}_j]) \supset \text{span}(\bar{\mathbf{A}}_1)$. Equivalently, $\text{null}(\bar{\mathbf{A}}_1) \supset \text{null}([\bar{\mathbf{A}}_1 \ \mathbf{A}_j])$. Denote $\dim(\text{null}(\bar{\mathbf{A}}_1)) = I - \alpha$ and $\dim(\text{null}([\bar{\mathbf{A}}_1 \ \mathbf{A}_j])) = I - \alpha - \beta_j$, with $\beta_j \geqslant 1$, $j = 1, \ldots, R$. Since the union of a countable number of subspaces of dimension $I - \alpha - \beta_j$ cannot cover a subspace of dimension $I - \alpha$, $\bigcup_{j=1}^{R} \text{null}([\bar{\mathbf{A}}_1 \ \mathbf{A}_j])$ does not cover $\text{null}(\bar{\mathbf{A}}_1)$. This implies that for a generic vector $\mathbf{x}_0$ in $\text{null}(\bar{\mathbf{A}}_1)$ we have

$$\omega'(\mathbf{x}_0^H\bar{\mathbf{A}}_1) = 0, \quad \omega'(\mathbf{x}_0^H\mathbf{A}_j) = 1, \quad j = 1, \ldots, R.$$

This means that for a generic vector $\mathbf{x}_0$ in $\text{null}(\bar{\mathbf{A}}_1)$ we have

$$\omega'(\mathbf{x}_0^H\bar{\mathbf{A}}) \leqslant R - 1 \leqslant R = \omega'(\mathbf{x}_0^H\mathbf{A}).$$

We have a contradiction with the condition in the lemma. Therefore, there exists a submatrix of $\mathbf{A}$, say, $\mathbf{A}_{j_0}$, such that $\bar{\mathbf{A}}_1 = \mathbf{A}_{j_0} \cdot \mathbf{L}$, in which $\mathbf{L}$ is square nonsingular.

We now construct a submatrix $\bar{\mathbf{A}}_0$ of $\bar{\mathbf{A}}$ by removing $\bar{\mathbf{A}}_1$ and a submatrix $\mathbf{A}_0$ of $\mathbf{A}$ by removing $\mathbf{A}_{j_0}$. Since for every vector $\mathbf{x}$, $\omega'(\mathbf{x}^H\bar{\mathbf{A}}_1) = \omega'(\mathbf{x}^H\mathbf{A}_{j_0})$ and, on the other hand, $\omega'(\mathbf{x}^H\mathbf{A}) \leqslant \omega'(\mathbf{x}^H\bar{\mathbf{A}})$ generically, we also have $\omega'(\mathbf{x}^H\mathbf{A}_0) \leqslant \omega'(\mathbf{x}^H\bar{\mathbf{A}}_0)$ generically. That is, $\mathbf{A}_0$ and $\bar{\mathbf{A}}_0$ satisfy the condition in the lemma, but they consist of only K submatrices. From the induction step we then have that $\bar{\mathbf{A}} = \mathbf{A} \cdot \mathbf{\Pi} \cdot \mathbf{\Lambda}$. This completes the proof. □

As mentioned above, the condition in Lemma 2.6 can be relaxed to the one in the equivalence lemma for partitioned matrices.

LEMMA 2.7. *Consider $\bar{\mathbf{A}}, \mathbf{A} \in \mathbb{K}^{I \times L}$, partitioned in the same but not necessarily uniform way into R submatrices that are full column rank. The following two statements are equivalent*:

(i) *For every $\mu \leqslant R$ there holds that for a generic vector $\mathbf{x}$ such that $\omega'(\mathbf{x}^H\bar{\mathbf{A}}) \leqslant \mu$, we have $\omega'(\mathbf{x}^H\mathbf{A}) \leqslant \omega'(\mathbf{x}^H\bar{\mathbf{A}})$.*

(ii) *For every $\mu \leqslant R - k'_{\bar{\mathbf{A}}} + 1$ there holds that for a generic vector $\mathbf{x}$ such that $\omega'(\mathbf{x}^H\bar{\mathbf{A}}) \leqslant \mu$, we have $\omega'(\mathbf{x}^H\mathbf{A}) \leqslant \omega'(\mathbf{x}^H\bar{\mathbf{A}})$.*

Proof. The implication of (ii) from (i) is trivial. The implication of (i) from (ii) is proved by contradiction.

Suppose there exists a nonzero vector $\mathbf{x}_0$ such that $\omega'(\mathbf{x}_0^H\mathbf{A}) > \omega'(\mathbf{x}_0^H\bar{\mathbf{A}})$ while $\omega'(\mathbf{x}_0^H\bar{\mathbf{A}}) > R - k'_{\bar{\mathbf{A}}} + 1$. Suppose that $\omega'(\mathbf{x}_0^H\bar{\mathbf{A}})$ is the smallest number bigger than $R - k'_{\bar{\mathbf{A}}} + 1$ for which (ii) does not hold, i.e., suppose that for every $\mu < \omega'(\mathbf{x}_0^H\bar{\mathbf{A}})$ there holds that for a generic vector $\mathbf{x}$ such that $\omega'(\mathbf{x}^H\bar{\mathbf{A}}) \leqslant \mu$, we have $\omega'(\mathbf{x}^H\mathbf{A}) \leqslant \omega'(\mathbf{x}^H\bar{\mathbf{A}})$. We can write

$$\omega'(\mathbf{x}_0^H\bar{\mathbf{A}}) = R - k'_{\bar{\mathbf{A}}} + \alpha \tag{2.2}$$

with $2 \leqslant \alpha < k'_{\bar{\mathbf{A}}}$ and

$$\omega'(\mathbf{x}_0^H \mathbf{A}) = R - k'_{\bar{\mathbf{A}}} + \alpha + \beta \tag{2.3}$$

with $1 \leqslant \beta < k'_{\bar{\mathbf{A}}} - \alpha$. Associated with $\mathbf{x}_0$, we have $k'_{\bar{\mathbf{A}}} - \alpha$ submatrices of $\bar{\mathbf{A}}$, say, $\bar{\mathbf{A}}_1, \ldots, \bar{\mathbf{A}}_{k'_{\bar{\mathbf{A}}}-\alpha}$, and $k'_{\bar{\mathbf{A}}} - \alpha - \beta$ submatrices of $\mathbf{A}$, say, $\mathbf{A}_1, \ldots, \mathbf{A}_{k'_{\bar{\mathbf{A}}}-\alpha-\beta}$, such that

$$\mathbf{x}_0 \in \text{null}([\bar{\mathbf{A}}_1 \ \ldots \ \bar{\mathbf{A}}_{k'_{\bar{\mathbf{A}}}-\alpha} \ ]) \cap \text{null}([\mathbf{A}_1 \ \ldots \ \mathbf{A}_{k'_{\bar{\mathbf{A}}}-\alpha-\beta}]).$$

$\mathbf{A}_1, \ldots, \mathbf{A}_{k'_{\bar{\mathbf{A}}}-\alpha-\beta}$ are the only submatrices of $\mathbf{A}$ of which the column space can possibly be contained in $\text{span}([\bar{\mathbf{A}}_1 \ \ldots \ \bar{\mathbf{A}}_{k'_{\bar{\mathbf{A}}}-\alpha} \ ])$. Otherwise, if there is one more submatrix, say, $\mathbf{A}_R$, of which the column space is contained in $\text{span}([\bar{\mathbf{A}}_1 \ \ldots \ \bar{\mathbf{A}}_{k'_{\bar{\mathbf{A}}}-\alpha}])$, then $\mathbf{x}_0^H \mathbf{A}_R = \mathbf{0}$ such that $\omega'(\mathbf{x}_0^H \mathbf{A}) = R - k'_{\bar{\mathbf{A}}} + \alpha + \beta - 1$, which contradicts (2.3).

Recall that by definition of $\omega'(\mathbf{x}_0^H \bar{\mathbf{A}})$ for every $\mu \leqslant R - k'_{\bar{\mathbf{A}}} + \alpha - 1 < \omega'(\mathbf{x}_0^H \bar{\mathbf{A}})$ there holds that for generic $\mathbf{x}$ such that $\omega'(\mathbf{x}^H \bar{\mathbf{A}}) \leqslant \mu$, we have $\omega'(\mathbf{x}^H \mathbf{A}) \leqslant \omega'(\mathbf{x}^H \bar{\mathbf{A}})$. Similar to Lemma 2.5, we can show that this implies that for every set of $c \geqslant k'_{\bar{\mathbf{A}}} - \alpha + 1$ submatrices of $\bar{\mathbf{A}}$, there exists a set of at least c submatrices of $\mathbf{A}$ such that span(matrix formed by these $c \geqslant k'_{\bar{\mathbf{A}}} - \alpha + 1$ submatrices of $\bar{\mathbf{A}}$) $\supseteq$ span(matrix formed by the c or more submatrices of $\mathbf{A}$).

Now we consider the matrices $[\bar{\mathbf{A}}_1 \ \ldots \ \bar{\mathbf{A}}_{k'_{\bar{\mathbf{A}}}-\alpha}]$ and $[\bar{\mathbf{A}}_1 \ \ldots \ \bar{\mathbf{A}}_{k'_{\bar{\mathbf{A}}}-\alpha} \ \bar{\mathbf{A}}_i]$, $i = k'_{\bar{\mathbf{A}}} - \alpha + 1, \ldots, R$. For each of these matrices we consider the submatrices of $\mathbf{A}$ of which the column space is contained in the column space of the given matrix.

First, recall that $\mathbf{A}_1, \ldots, \mathbf{A}_{k'_{\bar{\mathbf{A}}}-\alpha-\beta}$ are the only submatrices of $\mathbf{A}$ of which the column space is contained in $\text{span}([\bar{\mathbf{A}}_1 \ \ldots \ \bar{\mathbf{A}}_{k'_{\bar{\mathbf{A}}}-\alpha} \ ])$. Next, since $[\bar{\mathbf{A}}_1 \ \ldots \ \bar{\mathbf{A}}_{k'_{\bar{\mathbf{A}}}-\alpha} \ \bar{\mathbf{A}}_i]$ consists of $k'_{\bar{\mathbf{A}}} - \alpha + 1$ submatrices of $\bar{\mathbf{A}}$, there exist at least $k'_{\bar{\mathbf{A}}} - \alpha + 1$ submatrices $\mathbf{A}_{i_1}, \ldots, \mathbf{A}_{i_{k'_{\bar{\mathbf{A}}}-\alpha+1}}$ such that $\text{span}([\bar{\mathbf{A}}_1 \ \ldots \ \bar{\mathbf{A}}_{k'_{\bar{\mathbf{A}}}-\alpha} \ \bar{\mathbf{A}}_i]) \supseteq \text{span}([\mathbf{A}_{i_1} \ \ldots \ \mathbf{A}_{i_{k'_{\bar{\mathbf{A}}}-\alpha+1}}])$. Combining these results, we conclude that at least $\beta+1 = (k'_{\bar{\mathbf{A}}} - \alpha + 1) - (k'_{\bar{\mathbf{A}}} - \alpha - \beta)$ submatrices of $[\mathbf{A}_{i_1} \ \ldots \ \mathbf{A}_{i_{k'_{\bar{\mathbf{A}}}-\alpha+1}}]$, other than $\mathbf{A}_1, \ldots, \mathbf{A}_{k'_{\bar{\mathbf{A}}}-\alpha-\beta}$, have a column space that is in the span of $[\bar{\mathbf{A}}_1 \ \ldots \ \bar{\mathbf{A}}_{k'_{\bar{\mathbf{A}}}-\alpha} \ \bar{\mathbf{A}}_i]$. Denote by ϕ_i the set of those $\beta + 1$ or more submatrices of $[\mathbf{A}_{i_1} \ \ldots \ \mathbf{A}_{i_{k'_{\bar{\mathbf{A}}}-\alpha+1}}]$.

We prove that every two ϕ_i and ϕ_j are disjoint for $i \neq j$. Assume that a certain submatrix, say, $\mathbf{A}_j^i$, belongs to both ϕ_i and ϕ_j; then there exist matrices $\mathbf{X}$ and $\mathbf{Y}$ such that

$$\mathbf{A}_j^i = [\bar{\mathbf{A}}_1 \ \ldots \ \bar{\mathbf{A}}_{k'_{\bar{\mathbf{A}}}-\alpha} \bar{\mathbf{A}}_i] \cdot \mathbf{X} = [\bar{\mathbf{A}}_1 \ \ldots \ \bar{\mathbf{A}}_{k'_{\bar{\mathbf{A}}}-\alpha} \bar{\mathbf{A}}_j] \cdot \mathbf{Y}.$$

This, in turn, implies that there exists a matrix $\mathbf{Z}$ such that

$$[\bar{\mathbf{A}}_1 \ \ldots \ \bar{\mathbf{A}}_{k'_{\bar{\mathbf{A}}}-\alpha} \ \bar{\mathbf{A}}_i \ \bar{\mathbf{A}}_j] \cdot \mathbf{Z} = \mathbf{0}.$$

This is in contradiction with the definition of $k'_{\bar{\mathbf{A}}}$ and the fact that $\alpha \geqslant 2$.

Let us now count the number of submatrices of $\mathbf{A}$ in the above disjoint sets. In $\{\mathbf{A}_1, \ldots, \mathbf{A}_{k'_{\bar{\mathbf{A}}}-\alpha-\beta}\}$, there are $k'_{\bar{\mathbf{A}}} - \alpha - \beta$ submatrices. In each set ϕ_i there are at least $\beta + 1$ submatrices, and we have $R - k'_{\bar{\mathbf{A}}} + \alpha$ such ϕ_i. Therefore, the total number of submatrices of $\mathbf{A}$ from all disjoint sets is at least

$$k'_{\bar{\mathbf{A}}} - \alpha - \beta + (\beta + 1)(R - k'_{\bar{\mathbf{A}}} + \alpha) = \beta(R - k'_{\bar{\mathbf{A}}}) + R + (\alpha - 1)\beta,$$

which is strictly greater than R for $\alpha \geqslant 2$ and $\beta \geqslant 1$. Obviously, $\mathbf{A}$ has only R submatrices, so we have a contradiction. $\square$

3. Rank and k'-rank of Khatri–Rao products of partitioned matrices. In our analysis of the uniqueness of block decompositions [3], we make use of additional lemmas, besides the equivalence lemma for partitioned matrices, that establish certain Khatri–Rao products of partitioned matrices are full column rank. These are derived in the present section.

We start from a lemma that gives a lower-bound on the k-rank of a columnwise Khatri–Rao product. This lemma is proved in [8]. A shorter proof is given in [9, 10]. We give yet another proof, which is easier to generalize to Khatri–Rao products of arbitrarily partitioned matrices.

LEMMA 3.1. *Consider matrices* $\mathbf{A} \in \mathbb{K}^{I\times R}$ *and* $\mathbf{B} \in \mathbb{K}^{J\times R}$*.*

(i) *If* $k_{\mathbf{A}} = 0$ *or* $k_{\mathbf{B}} = 0$*, then* $k_{\mathbf{A}\odot_c\mathbf{B}} = 0$*.*

(ii) *If* $k_{\mathbf{A}} \geqslant 1$ *and* $k_{\mathbf{B}} \geqslant 1$*, then* $k_{\mathbf{A}\odot_c\mathbf{B}} \geqslant \min(k_{\mathbf{A}} + k_{\mathbf{B}} - 1, R)$*.*

Proof. First, we prove (i). If $k_{\mathbf{A}} = 0$, then $\mathbf{A}$ has an all-zero column. Consequently, $\mathbf{A} \odot_c \mathbf{B}$ also has an all-zero column and $k_{\mathbf{A}\odot_c\mathbf{B}} = 0$. The same holds if $k_{\mathbf{B}} = 0$. This completes the proof of (i).

Next, we prove (ii). Suppose $k_{\mathbf{A}} \geqslant 1$ and $k_{\mathbf{B}} \geqslant 1$. Let $m = \min(k_{\mathbf{A}} + k_{\mathbf{B}} - 1, R)$. We have to prove that any set of m columns of $\mathbf{A} \odot_c \mathbf{B}$ is linearly independent. Without loss of generality we prove that this is the case for the first m columns of $\mathbf{A} \odot_c \mathbf{B}$. (Another set of m columns can first be permuted to the first positions. This does not change the k-rank. We can then continue as below.) Let $\mathbf{A}_f = [\mathbf{a}_1 \dots \mathbf{a}_m]$, $\mathbf{B}_f = [\mathbf{b}_1 \dots \mathbf{b}_m]$, $\mathbf{A}_g = [\mathbf{a}_1 \dots \mathbf{a}_{k_{\mathbf{A}}}]$, $\mathbf{B}_g = [\mathbf{b}_{m-k_{\mathbf{B}}+1} \dots \mathbf{b}_m]$. Suppose $\mathbf{U} = (\mathbf{S}\mathbf{A}_f) \odot_c (\mathbf{T}\mathbf{B}_f) = (\mathbf{S} \otimes \mathbf{T})(\mathbf{A}_f \odot_c \mathbf{B}_f)$, where $\mathbf{S} \otimes \mathbf{T}$ is nonsingular if both $\mathbf{S}$ and $\mathbf{T}$ are nonsingular. Premultiplying a matrix by a nonsingular matrix does not change its rank nor its k-rank. Hence the rank of $\mathbf{U}$ is equal to the rank of $\mathbf{A}_f \odot_c \mathbf{B}_f$ if $\mathbf{S}$ and $\mathbf{T}$ are nonsingular. The same holds for the k-rank. We choose $\mathbf{S}$ and $\mathbf{T}$ in the following way:

$$\mathbf{S} = \begin{pmatrix} \mathbf{A}_g^{\dagger} \\ \mathbf{A}_g^{\dagger,\perp} \end{pmatrix} \quad \mathbf{T} = \begin{pmatrix} \mathbf{B}_g^{\dagger} \\ \mathbf{B}_g^{\dagger,\perp} \end{pmatrix} \tag{3.1}$$

in which $\mathbf{A}_g^{\dagger,\perp}$ is an (arbitrary) $((I - k_{\mathbf{A}}) \times I)$ matrix such that $\text{span}[(\mathbf{A}_g^{\dagger,\perp})^T] = \text{null}(\mathbf{A}_g)$, and in which $\mathbf{B}_g^{\dagger,\perp}$ is an (arbitrary) $((J - k_{\mathbf{B}}) \times J)$ matrix such that $\text{span}[(\mathbf{B}_g^{\dagger,\perp})^T] = \text{null}(\mathbf{B}_g)$. If we choose $\mathbf{S}$ and $\mathbf{T}$ this way, $\mathbf{U}$ has a very special structure.

Let us first illustrate this with an example. Assume a matrix $\mathbf{A} \in \mathbb{K}^{2\times 4}$ with $k_{\mathbf{A}} = 2$ and a matrix $\mathbf{B} \in \mathbb{K}^{3\times 4}$ with $k_{\mathbf{B}} = 3$. Then we have $\mathbf{A}_f = \mathbf{A}$, $\mathbf{B}_f = \mathbf{B}$, $k_{\mathbf{A}_f} = k_{\mathbf{A}}$ and $k_{\mathbf{B}_f} = k_{\mathbf{B}}$. We now have

$$\tilde{\mathbf{A}} = \mathbf{S} \cdot \mathbf{A}_f = \begin{pmatrix} 1 & 0 & \tilde{a}_{13} & \tilde{a}_{14} \\ 0 & 1 & \tilde{a}_{23} & \tilde{a}_{24} \end{pmatrix},$$

$$\tilde{\mathbf{B}} = \mathbf{T} \cdot \mathbf{B}_f = \begin{pmatrix} \tilde{b}_{11} & 1 & 0 & 0 \\ \tilde{b}_{21} & 0 & 1 & 0 \\ \tilde{b}_{31} & 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{U} = \tilde{\mathbf{A}} \odot_c \tilde{\mathbf{B}} = \begin{pmatrix} \tilde{b}_{11} & 0 & 0 & 0 \\ \tilde{b}_{21} & 0 & \tilde{a}_{13} & 0 \\ \tilde{b}_{31} & 0 & 0 & \tilde{a}_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \tilde{a}_{23} & 0 \\ 0 & 0 & 0 & \tilde{a}_{24} \end{pmatrix}.$$

Note that neither $\tilde{a}_{23}$ nor $\tilde{a}_{24}$ can be equal to zero, otherwise $k_{\tilde{\mathbf{A}}} < 2 = k_{\mathbf{A}_f}$ while $\mathbf{S}$ is nonsingular. On the other hand, $[\tilde{b}_{11}\ \tilde{b}_{21}\ \tilde{b}_{31}]$ cannot be equal to $[0\ 0\ 0]$, otherwise $k_{\tilde{\mathbf{B}}} = 0 < 3 = k_{\mathbf{B}_f}$ while $\mathbf{T}$ is nonsingular. We conclude that $\mathbf{U}$ is full column rank. Since $\mathbf{S}$ and $\mathbf{T}$ are nonsingular, $\mathbf{A}_f \odot_c \mathbf{B}_f$ is also full column rank.

In general, we have

$$\tilde{\mathbf{A}} = \mathbf{S} \cdot \mathbf{A}_f = \begin{pmatrix} \overbrace{\mathbf{I}_{k_\mathbf{A} \times k_\mathbf{A}}}^{k_\mathbf{A}} & \overbrace{\tilde{\mathbf{A}}(1 : k_\mathbf{A}, k_\mathbf{A}+1 : m)}^{m-k_\mathbf{A}} \\ \mathbf{0}_{(I-k_\mathbf{A}) \times k_\mathbf{A}} & \tilde{\mathbf{A}}(1+k_\mathbf{A} : I, k_\mathbf{A}+1 : m) \end{pmatrix},$$

$$\tilde{\mathbf{B}} = \mathbf{T} \cdot \mathbf{B}_f = \begin{pmatrix} \tilde{\mathbf{B}}(1 : k_\mathbf{B}, 1 : m-k_\mathbf{B}) & \mathbf{I}_{k_\mathbf{B} \times k_\mathbf{B}} \\ \underbrace{\tilde{\mathbf{B}}(k_\mathbf{B}+1 : J, 1 : m-k_\mathbf{B})}_{m-k_\mathbf{B}} & \underbrace{\mathbf{0}_{(J-k_\mathbf{B}) \times k_\mathbf{B}}}_{k_\mathbf{B}} \end{pmatrix}.$$

Key to understanding the structure of $\mathbf{U} = \tilde{\mathbf{A}} \odot_c \tilde{\mathbf{B}}$ is the specific form of the first $k_\mathbf{A}$ columns of $\tilde{\mathbf{A}}$ and the last $k_\mathbf{B}$ columns of $\tilde{\mathbf{B}}$, together with the fact that by definition of m, $m - k_\mathbf{B} < k_\mathbf{A}$ and $m - k_\mathbf{A} < k_\mathbf{B}$. This structure neatly generalizes the structure in the example above. The first $m - k_\mathbf{B}$ columns of $\mathbf{U}$ form a block-diagonal matrix, containing the first $m - k_\mathbf{B}$ columns of $\tilde{\mathbf{B}}$ in the diagonal blocks and zeros below. Each of the next $R - 2m + k_\mathbf{A} + k_\mathbf{B}$ columns of $\mathbf{U}$ is all-zero, except for a single 1 that is also the only nonzero entry of its row. The last $m - k_\mathbf{A}$ columns of $\mathbf{U}$ contain the corresponding entries of $\tilde{\mathbf{A}}(k_\mathbf{A} : I, k_\mathbf{A}+1 : m)$ in rows where they form the only nonzero entries. The columns of $\tilde{\mathbf{A}}(k_\mathbf{A} : I, k_\mathbf{A}+1 : m)$ cannot be all-zero. Suppose by contradiction that the nth column of $\tilde{\mathbf{A}}(k_\mathbf{A} : I, k_\mathbf{A}+1 : m)$ is all-zero. Then the first $k_\mathbf{A} - 1$ columns of $\tilde{\mathbf{A}}$, together with its $(k_\mathbf{A}+n)$th column, form a linearly dependent set. Hence, $k_{\tilde{\mathbf{A}}} < k_\mathbf{A} \leqslant k_{\mathbf{A}_f}$ while $\mathbf{S}$ is nonsingular. We have a contradiction. On the other hand, none of the first $m - k_\mathbf{B}$ columns of $\tilde{\mathbf{B}}$ can be all-zero either, otherwise $k_{\tilde{\mathbf{B}}} = 0 < k_\mathbf{B} \leqslant k_{\mathbf{B}_f}$ while $\mathbf{T}$ is nonsingular. We conclude that $\mathbf{U}$ is full column rank. Hence, $\mathbf{A}_f \odot_c \mathbf{B}_f$ is also full column rank. This completes the proof. ◻

Lemma 3.1 can be generalized to Khatri–Rao products of arbitrarily partitioned matrices as follows.

LEMMA 3.2. *Consider partitioned matrices* $\mathbf{A} = [\mathbf{A}_1\ \ldots\ \mathbf{A}_R]$ *with* $\mathbf{A}_r \in \mathbb{K}^{I \times L_r}$, $1 \leqslant r \leqslant R$, *and* $\mathbf{B} = [\mathbf{B}_1\ \ldots\ \mathbf{B}_R]$ *with* $\mathbf{B}_r \in \mathbb{K}^{J \times M_r}$, $1 \leqslant r \leqslant R$.

(i) *If* $k'_\mathbf{A} = 0$ *or* $k'_\mathbf{B} = 0$, *then* $k'_{\mathbf{A} \odot \mathbf{B}} = 0$.

(ii) *If* $k'_\mathbf{A} \geqslant 1$ *and* $k'_\mathbf{B} \geqslant 1$, *then* $k'_{\mathbf{A} \odot \mathbf{B}} \geqslant \min(k'_\mathbf{A} + k'_\mathbf{B} - 1, R)$.

Proof. We work in analogy with the proof of Lemma 3.1.

First, we prove (i). If $k'_\mathbf{A} = 0$, then $\mathbf{A}$ has a rank-deficient submatrix. Consequently, $\mathbf{A} \odot \mathbf{B}$ also has a rank-deficient submatrix and $k'_{\mathbf{A} \odot \mathbf{B}} = 0$. The same holds if $k'_\mathbf{B} = 0$. This completes the proof of (i).

Next, we prove (ii). Suppose $k'_\mathbf{A} \geqslant 1$ and $k'_\mathbf{B} \geqslant 1$. Let $m = \min(k'_\mathbf{A} + k'_\mathbf{B} - 1, R)$. We have to prove that any set of m submatrices of $\mathbf{A} \odot \mathbf{B}$ yields a linearly independent set of columns. Without loss of generality we prove that this is the case for the first m submatrices of $\mathbf{A} \odot \mathbf{B}$. Let $\mathbf{A}_f = [\mathbf{A}_1 \ldots \mathbf{A}_m]$, $\mathbf{B}_f = [\mathbf{B}_1 \ldots \mathbf{B}_m]$, $\mathbf{A}_g = [\mathbf{A}_1 \ldots \mathbf{A}_{k'_\mathbf{A}}]$, $\mathbf{B}_g = [\mathbf{B}_{m-k'_\mathbf{B}+1} \ldots \mathbf{B}_m]$. Suppose $\mathbf{U} = (\mathbf{S}\mathbf{A}_f) \odot (\mathbf{T}\mathbf{B}_f) = (\mathbf{S} \otimes \mathbf{T})(\mathbf{A}_f \odot \mathbf{B}_f)$. Hence the rank of $\mathbf{U}$ is equal to the rank of $\mathbf{A}_f \odot \mathbf{B}_f$ if $\mathbf{S}$ and $\mathbf{T}$ are nonsingular. The same holds for the k'-rank. We choose $\mathbf{S}$ and $\mathbf{T}$ as in (3.1). Let $\tilde{\mathbf{A}} = \mathbf{S} \cdot \mathbf{A}_f$ and $\tilde{\mathbf{B}} = \mathbf{T} \cdot \mathbf{B}_f$. The structure of $\mathbf{U}$ allows for a similar reasoning as in Lemma 3.1.

Let us first illustrate this with an example. Assume a matrix $\mathbf{A} \in \mathbb{K}^{4\times 6}$, consisting of 3 (4×2) submatrices, with $k'_{\mathbf{A}} = 2$, and a matrix $\mathbf{B} \in \mathbb{K}^{4\times 6}$, also consisting of three (4×2) submatrices, with $k'_{\mathbf{B}} = 2$. Then we have $\mathbf{A}_f = \mathbf{A}$, $\mathbf{B}_f = \mathbf{B}$, $k'_{\mathbf{A}_f} = k'_{\mathbf{A}}$, and $k'_{\mathbf{B}_f} = k'_{\mathbf{B}}$. We now have

$$\tilde{\mathbf{A}} = \mathbf{S} \cdot \mathbf{A}_f = \left(\begin{array}{cc|cc|cc} 1 & & & & \tilde{a}_{15} & \tilde{a}_{16} \\ & 1 & & & \tilde{a}_{25} & \tilde{a}_{26} \\ & & 1 & & \tilde{a}_{35} & \tilde{a}_{36} \\ & & & 1 & \tilde{a}_{45} & \tilde{a}_{46} \end{array}\right),$$

$$\tilde{\mathbf{B}} = \mathbf{T} \cdot \mathbf{B}_f = \left(\begin{array}{cc|cc|cc} \tilde{b}_{11} & \tilde{b}_{12} & 1 & & & \\ \tilde{b}_{21} & \tilde{b}_{21} & & 1 & & \\ \tilde{b}_{31} & \tilde{b}_{31} & & & 1 & \\ \tilde{b}_{41} & \tilde{b}_{41} & & & & 1 \end{array}\right),$$

$$\mathbf{U} = \tilde{\mathbf{A}} \odot \tilde{\mathbf{B}} = \left(\begin{array}{cccc|cccc|cccc} \tilde{b}_{11} & \tilde{b}_{12} & & & & & & & & & & \\ \tilde{b}_{21} & \tilde{b}_{22} & & & & & & & & & & \\ \tilde{b}_{31} & \tilde{b}_{32} & & & & & & & \tilde{a}_{15} & & \tilde{a}_{16} & \\ \tilde{b}_{41} & \tilde{b}_{42} & & & & & & & & \tilde{a}_{15} & & \tilde{a}_{16} \\ & & \tilde{b}_{11} & \tilde{b}_{12} & & & & & & & & \\ & & \tilde{b}_{21} & \tilde{b}_{22} & & & & & & & & \\ & & \tilde{b}_{31} & \tilde{b}_{32} & & & & & \tilde{a}_{25} & & \tilde{a}_{26} & \\ & & \tilde{b}_{41} & \tilde{b}_{42} & & & & & & \tilde{a}_{25} & & \tilde{a}_{26} \\ & & & & 1 & & & & & & & \\ & & & & & 1 & & & & & & \\ & & & & & & & & \tilde{a}_{35} & & \tilde{a}_{36} & \\ & & & & & & & & & \tilde{a}_{35} & & \tilde{a}_{36} \\ & & & & & & 1 & & & & & \\ & & & & & & & 1 & & & & \\ & & & & & & & & \tilde{a}_{45} & & \tilde{a}_{46} & \\ & & & & & & & & & \tilde{a}_{45} & & \tilde{a}_{46} \end{array}\right).$$

Note that $\tilde{\mathbf{A}}(3:4, 5:6)$ cannot be rank-deficient, otherwise $k'_{\tilde{\mathbf{A}}} < 2 = k'_{\mathbf{A}_f}$ while $\mathbf{S}$ is nonsingular. On the other hand, $\tilde{\mathbf{B}}(:, 1:2)$ cannot be rank-deficient, otherwise $k'_{\tilde{\mathbf{B}}} = 0 < 2 = k'_{\mathbf{B}_f}$ while $\mathbf{T}$ is nonsingular. We conclude that $\mathbf{U}$ is full column rank.

In general, the structure of $\mathbf{U}$ is as follows. Its leftmost $m - k'_{\mathbf{B}}$ submatrices form a block-diagonal matrix. The matrices in the diagonal blocks can be rank-deficient only if the corresponding submatrix of $\tilde{\mathbf{B}}$ is rank-deficient. This would imply that $k'_{\tilde{\mathbf{B}}} = 0 < k'_{\mathbf{B}_f}$ while $\mathbf{T}$ is nonsingular. Each column of the next $R - 2m + k'_{\mathbf{A}} + k'_{\mathbf{B}}$ submatrices of $\mathbf{U}$ is all-zero except for a single 1 that is also the only nonzero entry of its row. Consider the partitioning $\tilde{\mathbf{A}}(\sum_{r=1}^{k'_{\mathbf{A}}-1} L_r + 1 : I, \sum_{r=1}^{k'_{\mathbf{A}}} L_r + 1 : \sum_{r=1}^{m} L_r) = [\bar{\mathbf{A}}_{k'_{\mathbf{A}}+1} \dots \bar{\mathbf{A}}_m]$. The matrices $\bar{\mathbf{A}}_{k'_{\mathbf{A}}+1}, \dots, \bar{\mathbf{A}}_m$ can be rank-deficient only if $k'_{\tilde{\mathbf{A}}} < k'_{\mathbf{A}_f}$ while $\mathbf{S}$ is nonsingular. These matrices yield additional independent columns in $\mathbf{U}$. We conclude that $\mathbf{U}$ is full column rank. Hence, $\mathbf{A}_f \odot \mathbf{B}_f$ is full column rank. This completes the proof. □

Lemma 3.2 is a first tool that will be used in [3] to make sure that certain Khatri–Rao products of partitioned matrices are full column rank. Next, we generalize Lemma 2.2 in [2], saying that a columnwise Khatri–Rao product is generically full column rank, to Khatri–Rao products of arbitrarily partitioned matrices.

LEMMA 3.3. *Consider partitioned matrices* $\mathbf{A} = [\mathbf{A}_1 \ \ldots \ \mathbf{A}_R]$ *with* $\mathbf{A}_r \in \mathbb{K}^{I \times L_r}$, $1 \leqslant r \leqslant R$, *and* $\mathbf{B} = [\mathbf{B}_1 \ \ldots \ \mathbf{B}_R]$ *with* $\mathbf{B}_r \in \mathbb{K}^{J \times M_r}$, $1 \leqslant r \leqslant R$. *Generically we have that* $\text{rank}(\mathbf{A} \odot \mathbf{B}) = \min(IJ, \sum_{r=1}^{R} L_r M_r)$.

Proof. We prove the theorem by induction on R.

For $R = 1$, $\mathbf{A}_1$ and $\mathbf{B}_1$ are generically nonsingular. Hence, $\mathbf{A} \odot \mathbf{B} = \mathbf{A}_1 \otimes \mathbf{B}_1$ is generically nonsingular.

Now assume that the lemma holds for $R = 1, 2, \ldots, \tilde{R} - 1$. Then we prove that it also holds for $R = \tilde{R}$. Assume that $IJ \geqslant \sum_{r=1}^{\tilde{R}} L_r M_r$. A similar reasoning applies when $IJ > \sum_{r=1}^{\tilde{R}-1} L_r M_r$ but $IJ < \sum_{r=1}^{\tilde{R}} L_r M_r$. Let the columns of $\mathbf{A}_{\tilde{R}}^{\perp}$ form a basis for $\text{null}(\mathbf{A}_{\tilde{R}})$ and let the columns of $\mathbf{B}_{\tilde{R}}^{\perp}$ form a basis for $\text{null}(\mathbf{B}_{\tilde{R}})$. Define $\tilde{\mathbf{A}} = [\mathbf{A}_{\tilde{R}} \ \mathbf{A}_{\tilde{R}}^{\perp}]$ and $\tilde{\mathbf{B}} = [\mathbf{B}_{\tilde{R}} \ \mathbf{B}_{\tilde{R}}^{\perp}]$. Generically, $\mathbf{A}_{\tilde{R}}$ and $\mathbf{B}_{\tilde{R}}$ are full column rank. Hence, $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$, and $\tilde{\mathbf{A}} \otimes \tilde{\mathbf{B}}$ are also generically full column rank. Now replace the columns of $\mathbf{A}_{\tilde{R}} \otimes \mathbf{B}_{\tilde{R}}$ in $\tilde{\mathbf{A}} \otimes \tilde{\mathbf{B}}$ by random vectors $\mathbf{v}_j \in \mathbb{K}^{IJ}$, $j = 1, \ldots, L_{\tilde{R}} M_{\tilde{R}}$. Call the resulting matrix $\mathbf{C}$ and define $\mathbf{V} = [\mathbf{v}_1 \ \ldots \ \mathbf{v}_{L_{\tilde{R}} M_{\tilde{R}}}]$. For $\mathbf{C}$ to be rank deficient, a nontrivial linear combination of the columns of $[\mathbf{A}_{\tilde{R}}^{\perp} \otimes \mathbf{B}_{\tilde{R}} \ \ \mathbf{A}_{\tilde{R}} \otimes \mathbf{B}_{\tilde{R}}^{\perp} \ \ \mathbf{A}_{\tilde{R}}^{\perp} \otimes \mathbf{B}_{\tilde{R}}^{\perp}]$ must be in $\text{span}(\mathbf{V})$. This is a probability-zero event. Turned the other way around, if $\mathbf{v}_j \in \mathbb{K}^{IJ}$, $j = 1, \ldots, L_{\tilde{R}} M_{\tilde{R}}$ are a given linearly independent set of vectors and if we randomly choose $\mathbf{A}_{\tilde{R}} \in \mathbb{K}^{I \times L_{\tilde{R}}}$ and $\mathbf{B}_{\tilde{R}} \in \mathbb{K}^{J \times M_{\tilde{R}}}$, then the associated matrix $\mathbf{C}$ is full rank with probability one. Now let the vectors $\mathbf{v}_j$ be orthogonal to $\text{span}(\mathbf{A}_1 \otimes \mathbf{B}_1 \ \ldots \ \mathbf{A}_{\tilde{R}-1} \otimes \mathbf{B}_{\tilde{R}-1})$. Since the intersection of $\text{span}(\mathbf{V})$ and the orthogonal complement of $\mathbf{A}_{\tilde{R}} \otimes \mathbf{B}_{\tilde{R}}$ is generically zero, $\mathbf{V}^T(\mathbf{A}_{\tilde{R}} \otimes \mathbf{B}_{\tilde{R}})$ is generically full rank. In other words, $\mathbf{A}_{\tilde{R}} \otimes \mathbf{B}_{\tilde{R}}$ adds $L_{\tilde{R}} M_{\tilde{R}}$ independent directions to $[\mathbf{A}_1 \otimes \mathbf{B}_1 \ \ldots \ \mathbf{A}_{\tilde{R}-1} \otimes \mathbf{B}_{\tilde{R}-1}]$. Hence, $[\mathbf{A}_1 \otimes \mathbf{B}_1 \ \ldots \ \mathbf{A}_{\tilde{R}} \otimes \mathbf{B}_{\tilde{R}}]$ is generically full column rank. ∎

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